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CONSOB
COMMISSIONE NAZIONALE PER LE SOCIETÀ E LA BORSA

Risk Based Approach towards Transparency on Non-Equity Investment Products

Marcello Minenna – Head of Quantitative Analysis Unit, Consob

Columbia University
Department of Mathematics
Columbia University in the City of New York

Syllabus

Preliminaries

- regulatory framework
- products' risk-return profile VS investors' risk-return profile

Three-pillars approach

- financial structures
- 1st Pillar: unbundling and performance scenarios
  - return target products
    - unbundling
    - probabilistic performance scenarios
  - risk target and benchmark products
  - model risk assessment
- 2nd Pillar: the degree of risk
  - risk target and benchmark products
    - mapping
    - migration
  - return target products
- 3rd Pillar: recommended investment time horizon
  - risk target and benchmark products
    - first passage time
    - connection between probability, volatility and costs
    - characterization of the necessary condition in the space of returns
    - how to determine a consistent series of Time Horizons
  - return target products

Preliminaries

The transparency on the risk profile of non-equity investment products is based on three synthetic indicators (three pillars) – defined through the development of specific quantitative methods – in order to allow investors to take informed investment decisions.

~~Traditional narrative description of all possible risks associated with a financial product~~	Synthetic indicators robust, objective and backward verifiable

Preliminaries

Consob Annual Report 2008
Speech by the Chairman to the Financial Market

"The inclusion of indicators on performance scenarios, the degree of risk, costs and recommended investment time horizons in information documents will allow investors to assess and compare investments based on standard criteria.

This is a new approach on the international scene that meets the needs of a market, such as in Italy, where a high capacity for investment tends to privilege direct forms of investment".

Preliminaries

Consob Annual Report 2009
Speech by the Chairman to the Financial Market

"The weight of structured bonds on the total wealth of Italian families has been progressively increasing in the last decade …. This is a phenomenon that Consob is carefully monitoring, having considered the presence in retail investors portfolios of risky and illiquid bonds that do not offer an adequate return with respect to Government bonds yields."

Preliminaries

European Commission
The EU Single Market
Communication from the Commission on Packaged Retail Investment Products

The level of protection afforded to the retail investor should not vary according to the legal form of these products […]

This work:

- will provide a market (for packaged retail investment products) in which regulatory arbitrage does not drive savings towards particular products;
- has the objective to introduce a horizontal approach that will provide a coherent basis for the regulation of mandatory disclosures and selling practices at European level, irrespective of how the product is packaged or sold.

QdF Consob n. 63: A Quantitative Risk-Based Approach to the Transparency on Non-Equity Investment Products

Transparency regulation on the risk profile of non-equity investment products should be standard and translate into suitable regulatory provisions a coherent approach to risk measurement and to its correct representation to the potential investors.

This will create a context compatible with the concrete realization of a levelled playing field and with the prevention of any regulatory arbitrage which could arise due to the fragmentation of the current regulation.

[…] the only solution is represented by a thorough revision of both the European and the Italian regulatory framework in the direction of a single directive on the transparency for non-equity investment products.

Preliminaries

European Commission
The EU Single Market
Communication from the Commission on Packaged Retail Investment Products

Update on Commission work on
Packaged Retail Investment Products
16 december 2009

Pre-contractual disclosures

Common elements to allow for comparisons to include the structure of documents, order of sections, use of plain language, and focus on key information about nature of product, its risks, potential performance and costs.

QdF Consob n. 63: A Quantitative Risk-Based Approach to the Transparency on Non-Equity Investment Products

The regulatory choices Consob has made over time reflect its view of the prospectus as the privileged channels to realize an effective transparency both in the offering and in the distribution of non-equity investment products.

Such approach, developed and progressively implemented by Consob, is based on three pillars, corresponding to three synthetic indicators defined through the application of specific quantitative methods.

The three pillars fully define the contents of a product information sheet which should become the core of the prospectus and of the other transparency documentation intended to effectively.

Preliminaries

Proposal of the European Commission for a
DIRECTIVE OF THE EUROPEAN PARLIAMENT AND OF THE COUNCIL
amending Directive 2003/71/EC on the PROSPECTUS (September 2009)

Whereas (10):

"The summary of the prospectus is a key source of information for retail investors. It should be short, simple and easy for targeted investors to understand. It should focus on the key information that investors need in order to be able to make informed investment decisions. Its content should not be restricted to any predetermined number of words. The format and content of the summary should be determined in a way that ensures comparability with other investment products that are similar to the investment proposal described in the prospectus.".

FINANCIAL REGULATORY
REFORM: A NEW FOUNDATION

Protect consumers and investors from financial abuse.

To rebuild trust in our markets, we need strong and consistent regulation and supervision of consumer financial services and investment markets. …

We must promote transparency, simplicity, fairness, accountability, and access. We propose:

…

- Stronger regulations to improve the transparency, fairness, and appropriateness of consumer and investor products and services
- A level playing field and higher standards for providers of consumer financial products and services, whether or not they are part of a bank.

FINANCIAL REGULATORY
REFORM: A NEW FOUNDATION

Transparency.

We propose a new proactive approach to disclosure.

[…] all disclosures and other communications with consumers be reasonable: balanced in their presentation of benefits, and clear and conspicuous in their identification of costs, penalties, and risks.

Mandatory disclosure forms should be clear, simple, and concise.

Moreover, reasonableness does not mean a litany of every conceivable risk, which effectively obscures significant risks. It means identifying conspicuously the more significant risks. It means providing consumers with disclosures that help them to understand the consequences of their financial decisions.

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Three-pillars approach

- ❑ financial structures
- ❑ 1st Pillar: unbundling and performance scenarios
  - ➢ return target products
    - o unbundling
    - o probabilistic performance scenarios
  - ➢ risk target and benchmark products
  - ➢ model risk assessment
- ❑ 2nd Pillar: the degree of risk
  - ➢ risk target and benchmark products
    - o mapping
    - o migration
  - ➢ return target products
- ❑ 3rd Pillar: recommended investment time horizon
  - ➢ risk target and benchmark products
    - o first passage time
    - o connection between probability, volatility and costs
    - o characterization of the necessary condition in the space of returns
    - o how to determine a consistent series of Time Horizons
  - ➢ return target products

The implementation of the disclosure regulation on the risk-profile of non-equity investment products should allow the investor, even assisted by a financial advisor, to choose the financial product more suitable to his investment objectives.

Equity Mutual Funds
Liquidity Mutual Funds
Certificates
Bond Unit Linked
Covered Warrants
Structured Bond
Equity Unit Linked
BOND ETF
LIQUIDITY ETF
Bond
Bond Mutual Funds
EQUITY ETF
Index Linked
Structured ETF

Preliminaries: regulatory framework

Three different directives for the same financial engineering

UCITS Directive	Prospectus Directive	Life Assurance Directive
Equity Mutual Funds BOND ETF Liquidity Mutual Funds EQUITY ETF Structured ETF Bond Mutual Funds LIQUIDITY ETF	Certificates Bond Structured Bond Covered Warrants	Equity Unit Linked Bond Unit Linked Index Linked

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Preliminaries: products' risk-return profile VS investors' risk-return profile

The information to be provided to the investor, in a simple, clear and fair way, must allow an assessment of his needs in terms of:

Time goal:
liquidity/investment horizon
INVESTMENT HORIZON
(less than 3 years)

Equity Mutual Funds
Liquidity Mutual Funds
Bond Unit Linked
Structured Bond
BOND ETF
LIQUIDITY ETF
Equity Unit Linked
Bond
EQUITY ETF
Index Linked
Bond Mutual Funds
Structured ETF

Risk profile:
risk limit in terms of downside
RISKS
(medium-low)

Structured Bond
Index Linked
Liquidity Mutual Funds
Bond Unit Linked
LIQUIDITY ETF
Bond

Return goal:
target returns
RETURNS
(maximum return)

Bond
Liquidity Mutual Funds
LIQUIDITY ETF

Preliminaries: products' risk-return profile VS investors' risk-return profile

RETURNS | RISKS | INVESTMENT HORIZON

… allow the investor to match his needs with the features of the financial products and to make an informed investment decision

PREVENT MISBUYING

Preliminaries: products' risk-return profile VS investors' risk-return profile

INVESTOR NEEDS

INVESTMENT OBJECTIVES
- INVESTMENT HORIZON
- RISKS
- RETURNS

FINANCIAL SITUATION

EXPERIENCE AND KNOWLEDGE

MATCHING

INVESTMENT FIRMS

(MiFID suitability test)
Client Profiling

Investment firms interpret the needs of client according to their internal procedures that may differ from company to company

Potential Mismatch?

PRODUCT FEATURES

New Prospectus Directive

The format and content of the summary should be determined in a way that ensures comparability with other investment products that are similar to the investment proposal described in the prospectus.

KID
Key Investor Document
- INVESTMENT HORIZON
- RISKS
- RETURNS

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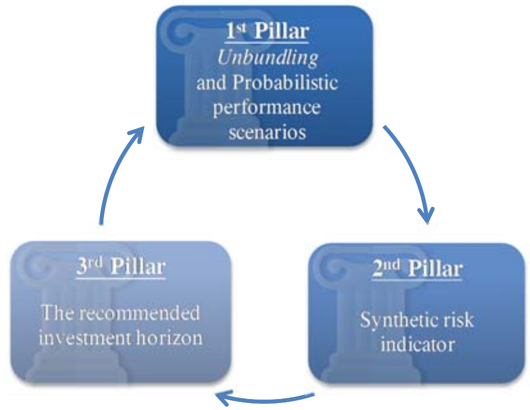
Three-pillars approach

RETURNS | RISKS | INVESTMENT HORIZON

The key qualitative information is made objective by using a three-pillars approach based on quantitative measures.

Three-pillars approach

The three pillars are closely linked together and offer to investors an organic and internally consistent representation of the risks, costs and potential performances of the product over the recommended investment horizon.



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Three-pillars approach: financial structures

The three-pillars approach is based on the preliminary classification of the products into three types of financial structures:

"Risk target" products

"Benchmark" products

"Return target" products

Three-pillars approach: financial structures

"Risk target" products

"Risk target" products invest in any market and any financial instrument in order to optimize over time a given target in terms of risk exposure.

"Benchmark" products

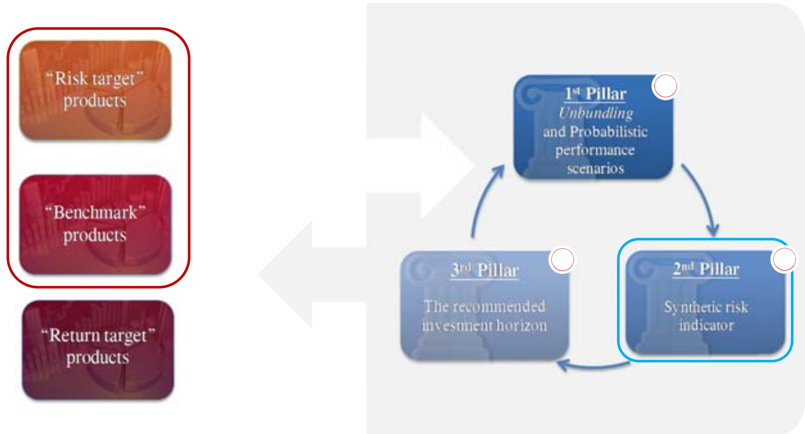
"Benchmark" products have an investment policy which is anchored to a benchmark, and in relation to this benchmark the asset management style may be either passive or active.

"Return target" products

"Return target" products feature a financial engineering (and, in some cases, a consequent investment policy) aimed at pursuing a minimum target return on the financial investment.

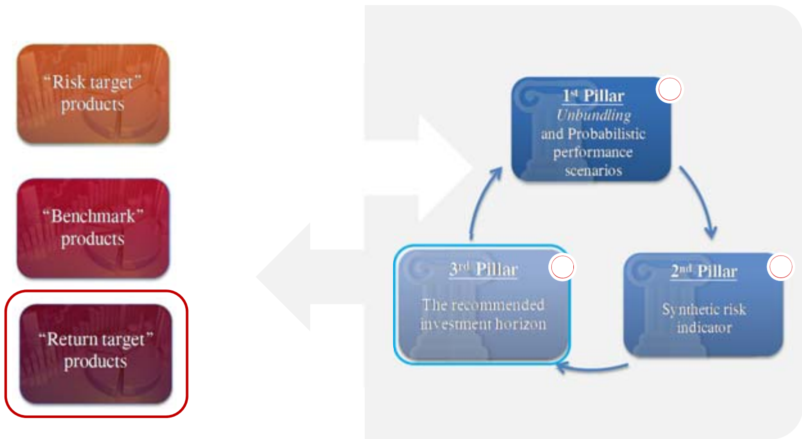
Three-pillars approach: financial structures

In "risk target" or "benchmark" products the degree of risk, together with the costs applied, allows to determine the recommended minimum investment time horizon. This horizon is used as the reference period to calculate the probability scenarios.



Three-pillars approach: financial structures

In "return target" products the target return at a given maturity clearly identifies the investment time horizon (a shorter holding period would compromise the liquidability of the product) w.r.t. which the probability scenarios and the degree of risk are determined.



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1st Pillar: *unbundling* and performance scenarios

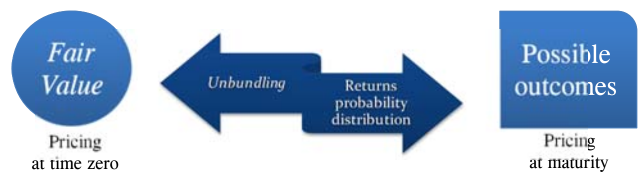
Unbundling and Probabilistic Performance Scenario

Performance risk
w.r.t. the risk-free asset
under the risk-neutral probability measure

... illustrates the unbundling of the price of the non-equity investment product at the time of subscription and provides a clear and concise information about its possible outcomes and costs.

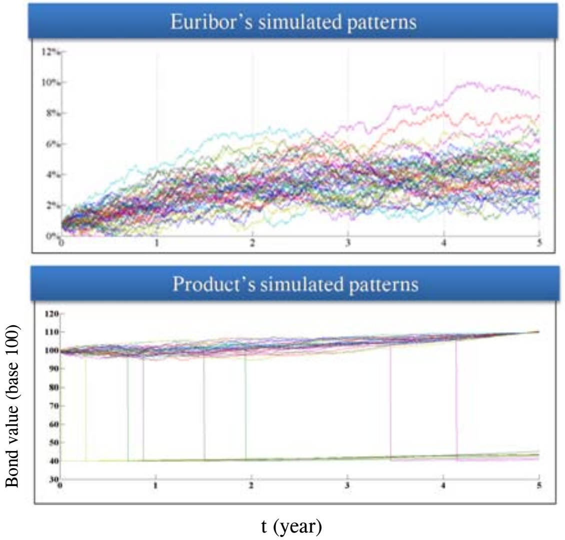
1st Pillar: return target products

In "return target" products (e.g. corporate bonds) the connection between the pricing at time zero and the pricing at maturity is evident, as the probability table is a necessary step to obtain the unbundling of the price of the product at time 0.



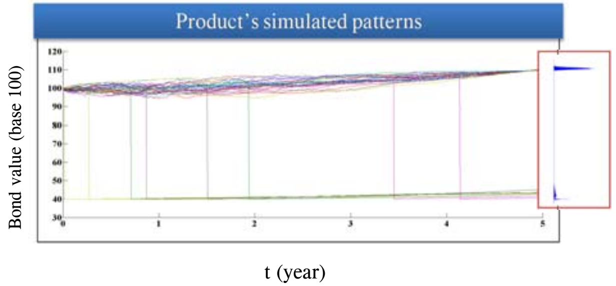
1st Pillar
Unbundling
and Probabilistic performance scenarios

5 year fixed-rate bond



1st Pillar
Unbundling
and Probabilistic performance scenarios

The final values of the bond at the end of the 5th year provide the probability distribution of potential returns (so-called *pricing* at maturity).



Possible outcomes
Pricing at maturity

1st Pillar
Unbundling
and Probabilistic performance scenarios

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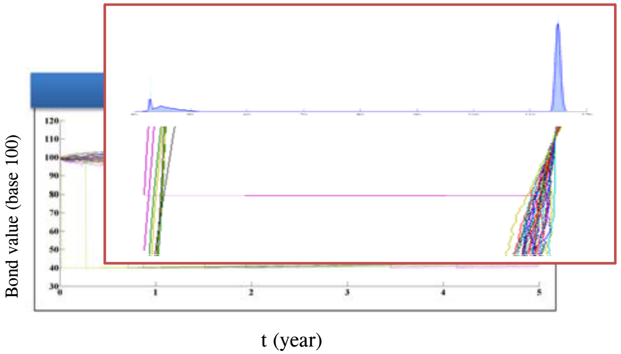

Possible outcomes
Pricing at maturity

1st Pillar
Unbundling
and Probabilistic performance scenarios

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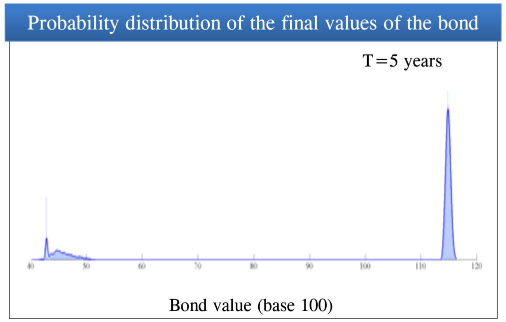

Possible outcomes
Pricing at maturity

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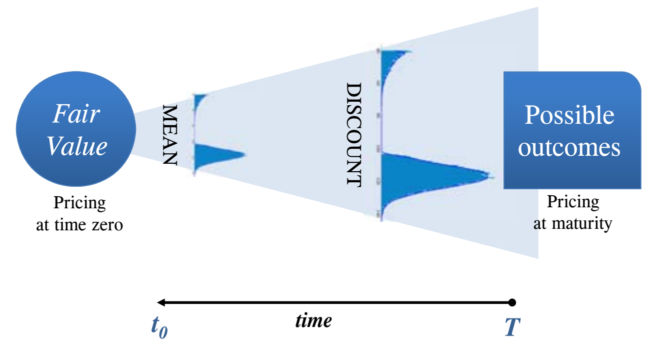
1st Pillar: return target products (*unbundling*)

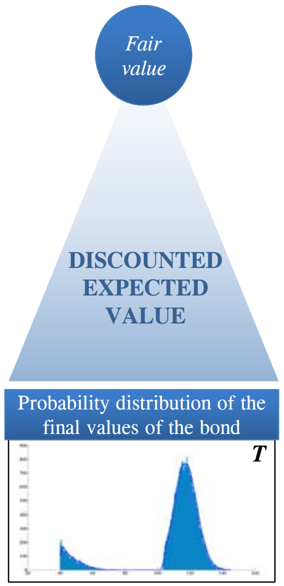
1st Pillar
Unbundling
and Probabilistic performance scenarios

The *unbundling* table shows the fair value of the product at time zero … which is equal to the expected value, under the risk-neutral probability measure, of the possible outcomes discounted at the risk-free rate.



1st Pillar: return target products (*unbundling*)

1st Pillar
Unbundling
and Probabilistic performance scenarios



1st Pillar: return target products (*unbundling*)

1st Pillar
Unbundling
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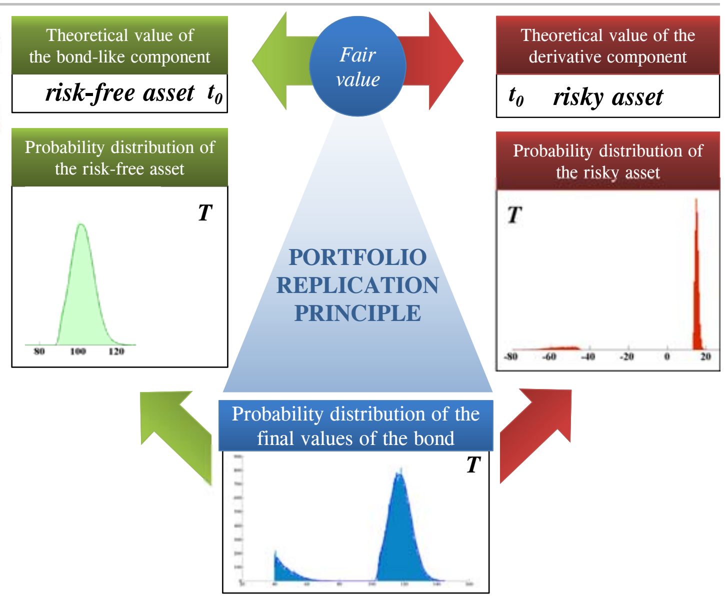

1st Pillar: return target products (*unbundling*)

1st Pillar
Unbundling
and Probabilistic performance scenarios

Theoretical value of the bond-like component
risk-free asset t_0

Fair value

Theoretical value of the derivative component
t_0 ***risky asset***

A	Theoretical value of the bond-like component	…
B	Theoretical value of the derivative component	…
C = A + B	*Fair value*	…
D	Explicit costs	…
E	Implicit costs	…
F = C + D + E	Issue price	100

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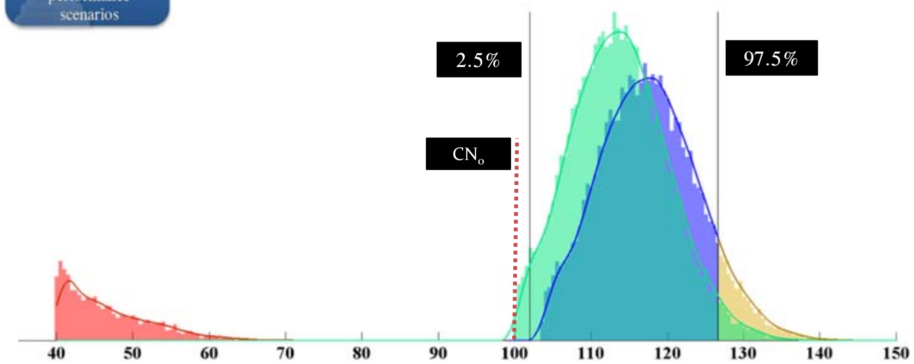
1st Pillar: return target products (probabilistic performance scenarios)

1st Pillar
Unbundling
and Probabilistic performance scenarios



The performance is negative

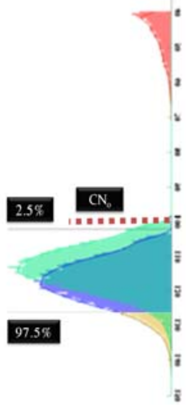
The performance is positive and lower than the risk-free asset

The performance is positive and in line with the risk-free asset

The performance is positive and higher than the risk-free asset

1st Pillar: return target products (probabilistic performance scenarios)

1st Pillar
Unbundling
and Probabilistic performance scenarios



SCENARIOS	PROBABILITY	MEDIAN VALUES
The performance is negative	%	€
The performance is positive but lower than the risk-free asset	%	€
The performance is positive and in line with the risk-free asset	%	€
The performance is positive and higher than the risk-free asset	%	€

1st Pillar: return target products (*unbundling* and performance scenarios)

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Connection between the pricing at time zero and the pricing at the end of recommended investment horizon

Time Zero

Financial investment table

A	Theoretical value of the bond-like component	…
B	Theoretical value of the derivative component	…
C = A + B	*Fair value*	…
D	Explicit costs	…
E	Implicit costs	…
F = C + D + E	Issue price	100

End of the recommended investment horizon

Table of probabilistic performance scenarios

SCENARIOS	PROBABILITY	MEDIAN VALUES
The performance is negative	%	€
The performance is positive but lower than the risk-free asset	%	€
The performance is positive and in line with the risk-free asset	%	€
The performance is positive and higher than the risk-free asset	%	€

1:1 Relationship

Syllabus

1st Pillar: risk target and benchmark products

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

"Risk target" products | "Benchmark" products | "Return target" products

In "risk target" and "benchmark" products, the above described connection between fair value and possible outcomes is satisfied at any time. In these products, the calculation of the returns' probability distribution is an intermediate step of the process carried out to determine the recommended minimum investment time horizon.

1st Pillar: risk target and benchmark products

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Connection between the pricing at time zero and the pricing at the end of recommended minimum investment horizon

Time Zero

Financial investment table

A	*Fair value*	…
B	Explicit costs	…
C	Implicit costs	…
D = A + B + E	Issue price	100

End of the recommended investment horizon

Table of probabilistic performance scenarios

SCENARIOS	PROBABILITY	MEDIAN VALUES
The performance is negative	%	€
The performance is positive but lower than the risk-free asset	%	€
The performance is positive and in line with the risk-free asset	%	€
The performance is positive and higher than the risk-free asset	%	€

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1st Pillar: model risk assessment

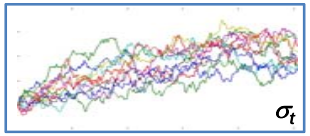
1st Pillar
Unbundling
and Probabilistic performance scenarios

Model Risk Assessment

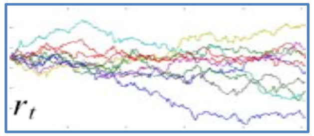
The recommended time horizon has a significant influence on the choice of the model

For Time Horizons greater than 1 year…..

I



II



1st Pillar: model risk assessment

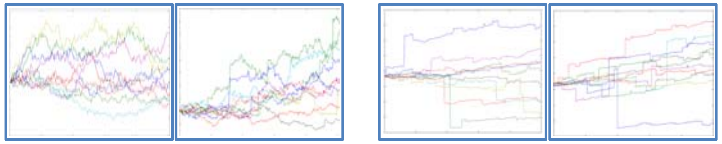
1st Pillar
Unbundling
and Probabilistic performance scenarios

Model Risk Assessment

The recommended time horizon has a significant influence on the choice of the model

Many possible choices…

I	II	III	IV
HESTON Stochastic Volatility Model	MERTON Jump Diffusion Model	CARR MADAN CHANG Variance Gamma Model	BARNDORFF NIELSEN NIG Model



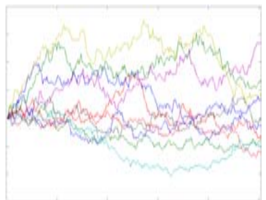
1st Pillar: model risk assessment

1st Pillar
Unbundling
and Probabilistic performance scenarios

Model Risk Assessment

Different Hypothesis on the stochastic processes of the underlyings can be made in order to capture the markets complexities

HESTON
Stochastic Volatility Model



$$\begin{cases} dS_t = rS_t dt + \sigma_t S_t dW_t^{(S)} \\ d\sigma_t^2 = \kappa\left(\theta - \sigma_t^2\right)dt + \zeta\sigma_t dW_t \end{cases}$$

Variance as a diffusive process

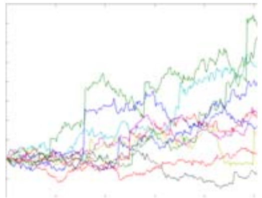
Complex to calibrate

1st Pillar
Unbundling
and Probabilistic performance scenarios

Model Risk Assessment

Different Hypothesis on the stochastic processes of the underlyings can be made in order to capture the markets complexities

MERTON
Jump Diffusion Model



$$dS_t = (r-\lambda\mu)S_t dt + \sigma S_t dW_t + J_t S_t dN_t$$

Able to replicate abrupt movements of the underlying

Constant Volatility Hypothesis

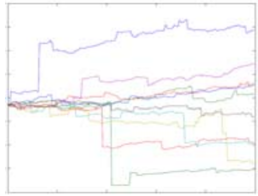
1st Pillar: model risk assessment

1st Pillar
Unbundling
and Probabilistic performance scenarios

Model Risk Assessment

Different Hypothesis on the stochastic processes of the underlyings can be made in order to capture the markets complexities

CARR MADAN CHANG
Variance Gamma Model



$$\begin{cases} S_t = S_0 e^{rt+\omega t+VG_t} \\ VG_t = \theta t + \sigma W_{G_t} \end{cases}$$

Normal Variance Mean mixture with a Gamma subordinator

Stochastic Time Hypothesis

Straightforward to calibrate

1st Pillar: model risk assessment

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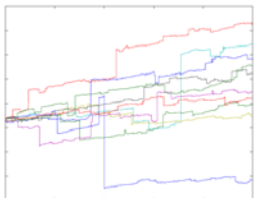
Unbundling
and Probabilistic performance scenarios

Model Risk Assessment

Different Hypothesis on the stochastic processes of the underlyings can be made in order to capture the markets complexities

BARNDORFF NIELSEN
NIG Model



$$\begin{cases} S_t = S_0 e^{\xi t+NIG_t} \\ NIG_t = \mu t + \beta IG_t + IG_t \cdot W_t \end{cases}$$

Normal Variance Mean mixture with an Inverse Gaussian subordinator

Semi-heavy tails

Great flexibility in calibrating the shape of probability density

1st Pillar: model risk assessment

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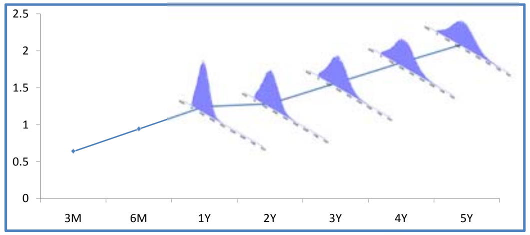
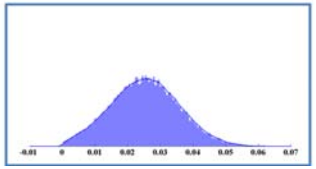
Unbundling
and Probabilistic performance scenarios

Step 1: Calculation of the Probability Distribution of the Notional Capital at the end of recommended time horizon





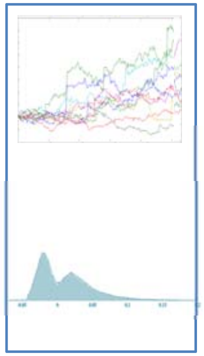
1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

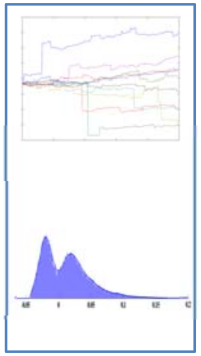
Step 2: Calculation of the Probability Distribution of the Invested Capital at the end of recommended time horizon

Heston

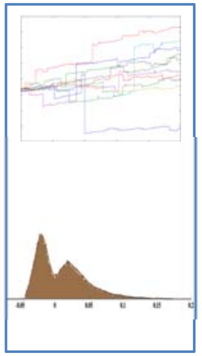
Merton



V G



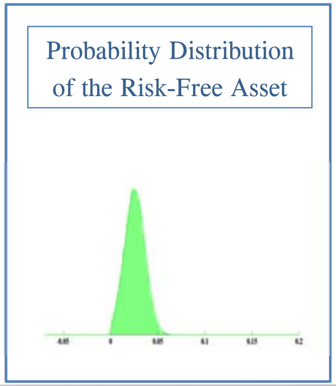
NIG

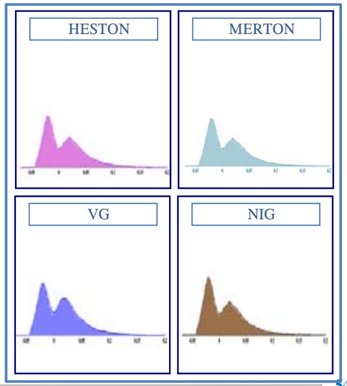


1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

Step 2: Calculation of the Probability Distribution of the Invested Capital at the end of recommended time horizon





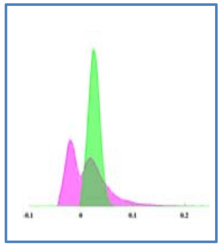
1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

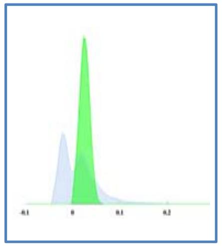
Step 3: Probabilistic comparison with the Risk-Free Asset

Analysing the probability distributions…

Heston



Merton



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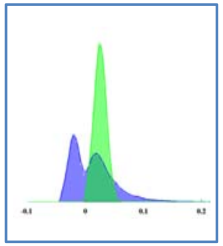

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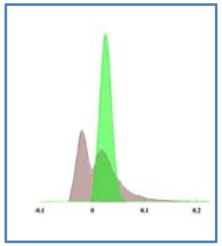

1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

… the following output is obtained:

Heston

Scenarios	Probability	Median Values
The performance is negative	46.61 %	€ 90.50
The performance is positive but lower than the risk-free asset	3.39%	€ 101.26
The performance is positive and in line with the risk-free asset	33.28 %	€ 112.19
The performance is positive and higher than the risk-free asset	16.72 %	€ 139.93

Merton

Scenarios	Probability	Median Values
The performance is negative	42.695 %	€ 89.26
The performance is positive but lower than the risk-free asset	4.74%	€ 102.54
The performance is positive and in line with the risk-free asset	35.7%	€ 110.09
The performance is positive and higher than the risk-free asset	16.86 %	€ 142.65

V G

Scenarios	Probability	Median Values
The performance is negative	43.91 %	€ 91.25
The performance is positive but lower than the risk-free asset	5.23%	€ 102.1
The performance is positive and in line with the risk-free asset	36.8%	€ 109.24
The performance is positive and higher than the risk-free asset	14.06 %	€ 141.77

NIG

Scenarios	Probability	Median Values
The performance is negative	48.1%	€ 93.40
The performance is positive but lower than the risk-free asset	2.6%	€ 101.91
The performance is positive and in line with the risk-free asset	34.28 %	€ 114.23
The performance is positive and higher than the risk-free asset	15.02 %	€ 142.13

1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: |Δ| < 4.7%

Heston

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1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: |Δ| < 2.7%

Heston

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1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: |Δ| < 3.7%

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1st Pillar: model risk assessment

1st Pillar
Unbundling and Probabilistic performance scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: |Δ| < 1.2%

Heston

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Syllabus

Preliminaries
- regulatory framework
- products' risk-return profile VS investors' risk-return profile

Three-pillars approach
- financial structures
- 1st Pillar: unbundling and performance scenarios
  - return target products
    - unbundling
    - probabilistic performance scenarios
  - risk target and benchmark products
  - model risk assessment
- 2nd Pillar: the degree of risk
  - risk target and benchmark products
    - mapping
    - migration
  - return target products
- 3rd Pillar: recommended investment time horizon
  - risk target and benchmark products
    - first passage time
    - connection between probability, volatility and costs
    - characterization of the necessary condition in the space of returns
    - how to determine a consistent series of Time Horizons
  - return target products

2nd Pillar
Synthetic risk indicator

Synthetic Risk Indicator

… provides a description, on a qualitative scale, of the risk level of the financial products based on volatility measures.

… represents in an explicit way the riskiness of the product embedded in the probabilistic performance scenarios of the first pillar.

2nd Pillar
Synthetic risk indicator

"Risk target" products | "Benchmark" products | "Return target" products

The degree of risk of "risk target" and "benchmark" products is initially identified by the intermediary choosing the risk class which he deems to better match the specific features of the product's financial engineering over the recommended investment time horizon.

During this horizon, the intermediary monitor any possible migration of the degree of risk to a different risk class or, for "benchmark" products, to a different management class (i.e. the intensity of the asset management activity in terms of deviation from the chosen benchmark).

2nd Pillar
Synthetic risk indicator

Synthetic Risk Indicator
(The degree of risk)

Six qualitative risk classes

RISK

Low
Medium-Low
Medium
Medium-High
High
Very High

2nd Pillar
Synthetic risk indicator

Synthetic Risk Indicator
(The degree of risk)

Time evolution of the volatility

↓

Mapping of the qualitative risk classes into corresponding volatility intervals

Syllabus

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

The mapping is performed according to the following steps:

↓

Step 1: Definition of Loss Intervals

Step 2: Mapping of Loss Intervals to the corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 1: Definition of Loss Intervals

What is a loss in a financial investment?

RISK NEUTRALITY PRINCIPLE

$$\text{LOSS} \in (-100\%, \overline{r^{rf}}\,]$$

$\overline{r^{rf}}$ = average of the probability distribution of the risk-free rate

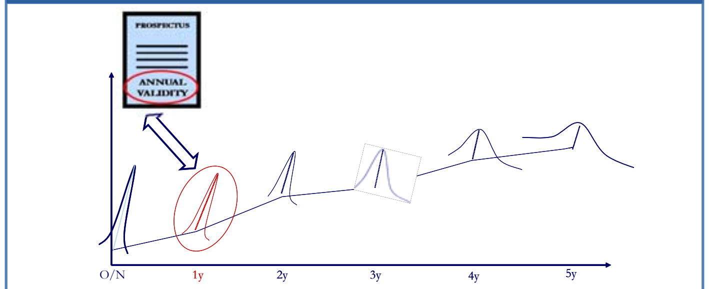
2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 1: Definition of Loss Intervals

given the risk-free yield curve and the associated volatility surface…

Risk-free yield curve and volatility surface



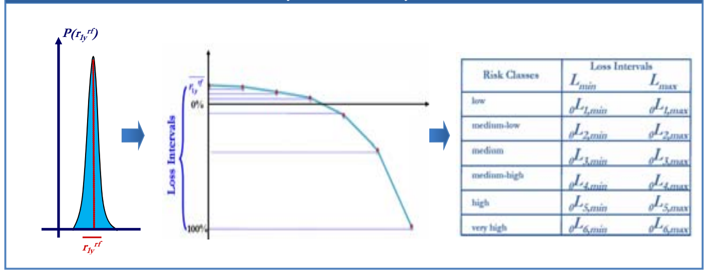
2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 1: Definition of Loss Intervals

the corresponding annual loss interval (multiple of r_{1y}^{rf} according to an exponential function) is associated to each risk class

From risk-free yield curve to one year loss intervals



2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 2: Mapping into Initial Volatility Intervals

Risk Classes	Loss Intervals L_{min}	L_{max}
low	$_0L_{1,min}$	$_0L_{1,max}$
medium-low	$_0L_{2,min}$	$_0L_{2,max}$
medium	$_0L_{3,min}$	$_0L_{3,max}$
medium-high	$_0L_{4,min}$	$_0L_{4,max}$
high	$_0L_{5,min}$	$_0L_{5,max}$
very high	$_0L_{6,min}$	$_0L_{6,max}$

Risk Classes	Volatility Intervals σ_{min}	σ_{max}
low	$_0\sigma_{1,min}$	$_0\sigma_{1,max}$
medium-low	$_0\sigma_{2,min}$	$_0\sigma_{2,max}$
medium	$_0\sigma_{3,min}$	$_0\sigma_{3,max}$
medium -high	$_0\sigma_{4,min}$	$_0\sigma_{4,max}$
high	$_0\sigma_{5,min}$	$_0\sigma_{5,max}$
very high	$_0\sigma_{6,min}$	$_0\sigma_{6,max}$

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals

TOOLS

√ GARCH Diffusive Models

√ Non linear Stochastic Programming

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

The Weak Convergence Theorem on R^2

The jump-continuous process $\{X_t^h\}$, whose measurable space is $(\mathbb{R}^2, \mathbb{B}(\mathbb{R}^2))$, converges weakly for h↓ 0 to the continuous process $\{X_t\}$ which has a unique distribution and is characterized by the following stochastic differential equation:

$$dX_t = b(x,t)dt + \sigma(x,t)dW_{2,t}$$

where $W_{2,t}$ is a two-dimensional standard Brownian motion, if the conditions 1-4 hereafter are satisfied.

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

Condition 1

If ∃ a δ>0 s.t. $\lim_{h\downarrow 0}\begin{pmatrix} c_{h,\delta}(x_1,t) \\ c_{h,\delta}(x_2,t)\end{pmatrix} = 0$ then ∃ $a(x,t)$ and $b(x,t)$ s.t.:

$$\lim_{h\downarrow 0}\begin{pmatrix} b_h(x_1,t) \\ b_h(x_2,t)\end{pmatrix} = \begin{pmatrix} b(x_1,t) \\ b(x_2,t)\end{pmatrix}$$

$$\lim_{h\downarrow 0}\begin{pmatrix} a_h(x_1,t) & a_h((x_1,x_2),t) \\ a_h((x_2,x_1),t) & a_h(x_2,t)\end{pmatrix} = \begin{pmatrix} a(x_1,t) & 0 \\ 0 & a(x_2,t)\end{pmatrix}$$

Condition 2

∃ $\sigma(x,t)$ s.t.: $\forall x_1 \in \mathbb{R}^1, \forall x_2 \in \mathbb{R}^1$ then $\begin{pmatrix} \sigma(x_1,t) & 0 \\ 0 & \sigma(x_2,t)\end{pmatrix} = \begin{pmatrix} \sqrt{a(x_1,t)} & 0 \\ 0 & \sqrt{a(x_2,t)}\end{pmatrix}$

Condition 3

For h↓ 0, X_0^h converges in distribution to a random variable X_0 with probability measure v_0 on $(\mathbb{R}^2, \mathbb{B}(\mathbb{R}^2))$

Condition 4

v_0, $a(x,t)$ and $b(x,t)$ uniquely specify the distribution of the process $\{X_t\}$ characterized by an initial distribution v_0, a conditional second moment $a(x,t)$ and a conditional first moment $b(x,t)$

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

The Continuous Limit of the M-GARCH(1,1)

statement

from the M-GARCH(1,1)

$$\begin{cases} X_k - X_{k-1} = \gamma \cdot (\eta - X_{k-1}) + \sigma_k \tilde{Z}_k \\ \text{and} \\ \ln \sigma_{k+1}^2 - \ln \sigma_k^2 = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln \sigma_k^2 + \beta_1^{(k)} \ln Z_k^2 \\ \text{or, equivalently} \\ \ln \sigma_{k+1}^2 - \ln \sigma_k^2 = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln \sigma_k^2 + 2\beta_1^{(k)} \ln |Z_k| \end{cases}$$

$\tilde{Z}_k$ and Z_k are i.i.d $N(0,1)$

Weak Convergence theorem

$$dX_t = q(\mu - X_t)dt + \sigma_t dW_t$$

$$d\ln \sigma_t^2 = \left(\beta_0 + 2\beta_1 E(\ln|Z_t|) + (\beta_1 - 1)\ln \sigma_t^2\right)dt + 2|\beta_1|\sqrt{Var(\ln|Z_t|)}dW_t^*$$

Z_t is $N(0,1)$

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

The Prediction Interval for the Volatility

key point

then

From the Diffusion Limit of the M-GARCH(1,1) Process it is possible to establish a *Predictive Interval for* σ_t

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

The Prediction Interval for the Volatility

distributional properties of the S.D.E. of the M-GARCH(1,1)

$$d\ln\sigma_t^2 = \left[\beta_0 + 2\beta_1\mathbf{E}(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2\right]dt + 2|\beta_1|\sqrt{Var(\ln|Z_t|)}dW_t^*$$

O.U. Process

$$\ln\sigma_t^2 \sim N\left(\left(\ln\sigma_s^2 + \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)(t-s)} - \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)};\ \sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2\left(e^{2(\beta_1-1)(t-s)}-1\right)}{2(\beta_1-1)}}\right)$$

2nd Pillar: risk target and benchmark products

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

matching of the first two conditional moments

discrete process

$$E\left(\ln\sigma_k^2\right) = \beta_0^{(k)} + \beta_1^{(k)}\ln\sigma_{k-1}^2 + 2\beta_1^{(k)}E\left(\ln|Z_{k-1}|\right)$$

$$Var\left(\ln\sigma_k^2\right) = 4\left(\beta_1^{(k)}\right)^2 Var\left(\ln|Z_{k-1}|\right)$$

2nd Pillar: risk target and benchmark products

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

matching of the first two conditional moments

continuous process

$$E\left(\ln\sigma_t^2\right) = \left(\ln\sigma_{t-1}^2 + \frac{\beta_0 + 2\beta_1 E\left(\ln|Z_t|\right)}{(\beta_1 - 1)}\right)e^{(\beta_1-1)} - \frac{\beta_0 + 2\beta_1 E\left(\ln|Z_t|\right)}{(\beta_1 - 1)}$$

$$Var\left(\ln\sigma_t^2\right) = \frac{4\beta_1^2 Var\left(\ln|Z_t|\right)}{2(\beta_1 - 1)}\left(e^{2(\beta_1-1)} - 1\right)$$

2nd Pillar: risk target and benchmark products

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

matching of the first two conditional moments

matching of the parameters

$$\left|\beta_1^{(k)}\right| = |\beta_1|\sqrt{\frac{e^{2(\beta_1-1)} - 1}{2(\beta_1 - 1)}}$$

$$\beta_0^{(k)} = -2|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}}E\left(\ln|Z_{k-1}|\right) - |\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}}\ln\sigma_{k-1}^2 + e^{(\beta_1-1)}\ln\sigma_{k-1}^2 + \frac{[\beta_0+2\beta_1 E(\ln|Z_{k-1}|)]\left(e^{(\beta_1-1)}-1\right)}{\beta_1-1}$$

2nd Pillar

Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

matching of the first two conditional moments

the discrete process can be written as:

$$\ln\sigma_k^2 - \ln\sigma_{k-1}^2 = \frac{[\beta_0 + 2\beta_1 E(\ln|Z_{k-1}|)]\left(e^{(\beta_1-1)}-1\right)}{\beta_1-1} - 2|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}}E(\ln|Z_{k-1}|) + \left(e^{(\beta_1-1)}-1\right)\ln\sigma_{k-1}^2 + 2|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}}\ln|Z_{k-1}|$$

2nd Pillar

Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

maximum likelihood estimation

setting

$$y_k := \ln\sigma_k^2 - \ln\sigma_{k-1}^2$$

$$\varepsilon := \ln|Z_{k-1}|$$

2nd Pillar

Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

maximum likelihood estimation

then

$$y_k = \frac{(\beta_0 - 1.27\beta_1)\left(e^{(\beta_1-1)}-1\right)}{\beta_1-1} + 1.27|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}} + \left(e^{(\beta_1-1)}-1\right)\ln\sigma_{k-1}^2 + 2|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}}\varepsilon$$

where we used: $E(\ln|Z_{k-1}|) = -0.6351$

2nd Pillar

Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

maximum likelihood estimation

the likelihood function

$$L(w;\underline{\beta}) = \prod_{k=2}^{K}\left[\frac{1}{|\beta_1|\sqrt{2\pi}}\sqrt{\frac{2(\beta_1-1)}{e^{2(\beta_1-1)}-1}}\cdot e^{\left(\frac{1}{2|\beta_1|}\sqrt{\frac{2(\beta_1-1)}{e^{2(\beta_1-1)}-1}}\cdot w_k\right)} \cdot e^{\left(-\frac{1}{2}\exp\left(\frac{1}{|\beta_1|}\sqrt{\frac{2(\beta_1-1)}{e^{2(\beta_1-1)}-1}}\cdot w_k\right)\right)}\right]$$

where:

$$\underline{\beta} := (\beta_0, \beta_1)$$

$$w_k := y_k - \frac{(\beta_0 - 1.27\beta_1)\left(e^{(\beta_1-1)}-1\right)}{\beta_1-1} - 1.27|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}} - \left(e^{(\beta_1-1)}-1\right)\ln\sigma_{k-1}^2$$

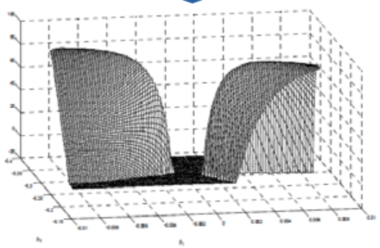
2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

maximum likelihood estimation

shape of the associated log-likelihood



2nd Pillar: risk target and benchmark products

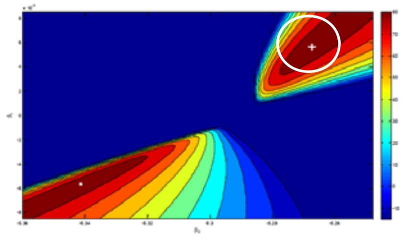
2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

maximum likelihood estimation

β_0 and β_1 estimates



2nd Pillar: risk target and benchmark products

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models

the estimated parameters enter in the bounds of the volatility prediction interval

$$\sigma^G_{t,\min} = e^{\frac{-z_{\frac{1+\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2\left(e^{2(\beta_1-1)}-1\right)}{2(\beta_1-1)}}+\left(\ln\sigma^2_{t-1}+\frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}}{2}}$$

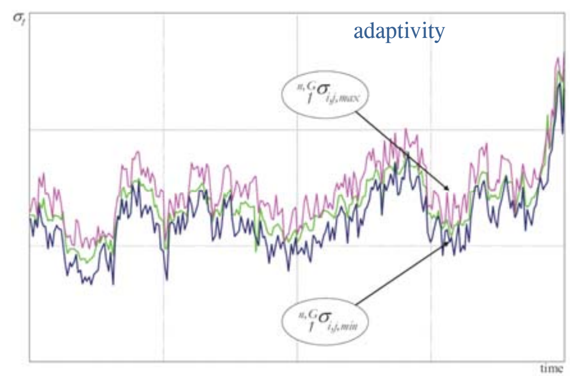
$$\sigma^G_{t,\max} = e^{\frac{z_{\frac{1+\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2\left(e^{2(\beta_1-1)}-1\right)}{2(\beta_1-1)}}+\left(\ln\sigma^2_{t-1}+\frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}}{2}}$$

2nd Pillar: risk target and benchmark products

2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

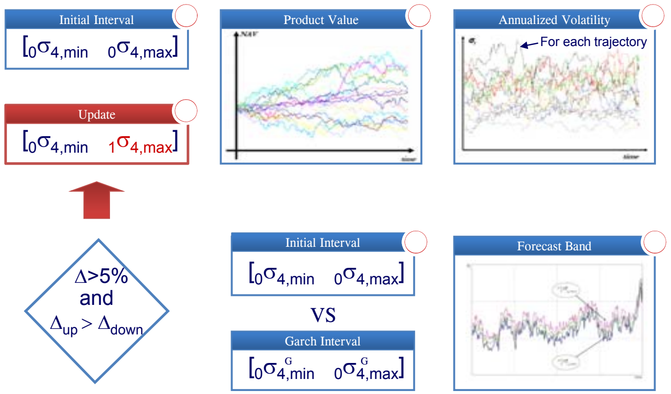
Step 3: Fine-tuning of Volatility Intervals: GARCH Diffusive Models



2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

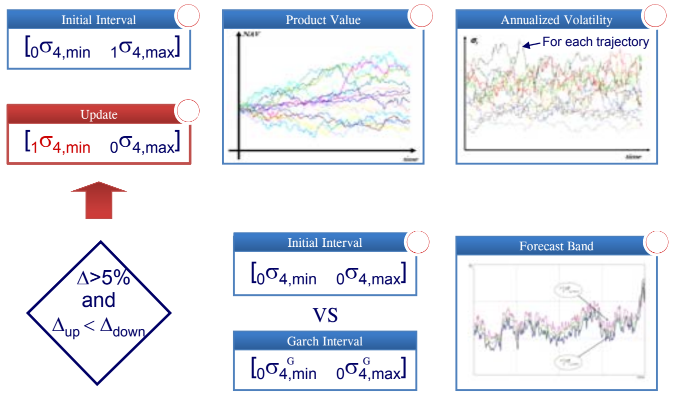
Step 3: Fine-tuning of Volatility Intervals: Non Linear Stochastic Programming



2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

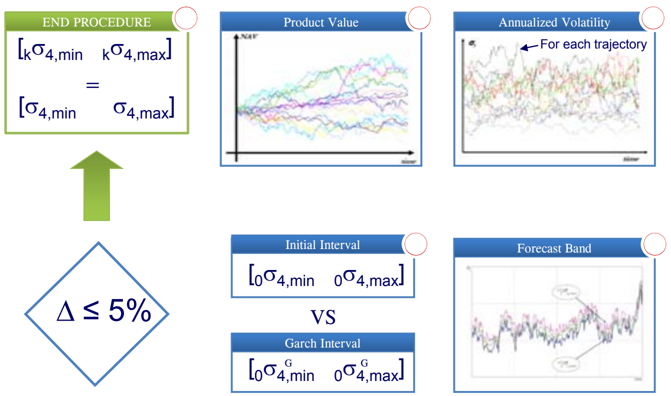
Step 3: Fine-tuning of Volatility Intervals: Non Linear Stochastic Programming



2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals: Non Linear Stochastic Programming



2nd Pillar
Synthetic risk indicator

Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 3: Fine-tuning of Volatility Intervals

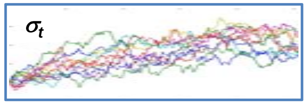
OUTPUT

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
Low	0.01%	0.49%
Medium-Low	0.50%	1.59%
Medium	1.60%	3.99%
Medium-High	4.00%	9.99%
High	10.00%	24.99%
Very High	25.00%	>25.00%

2nd Pillar
Synthetic risk indicator

Synthetic Risk Indicator
(The degree of risk)

Time evolution of the volatility



Mapping of the qualitative risk classes into corresponding volatility intervals

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
Low	0.01%	0.49%
Medium-Low	0.50%	1.59%
Medium	1.60%	3.99%
Medium-High	4.00%	9.99%
High	10.00%	24.99%
Very High	25.00%	>25.00%

2nd Pillar
Synthetic risk indicator

"Risk target" products | "Benchmark" products | "Return target" products

For benchmark products the degree of risk is supplemented by a synthetic indicator of the asset management style:

passive or active

In this second case, the intensity of the active management style depends on the extent of the deviation from the benchmark and on its direction

2nd Pillar
Synthetic risk indicator

Synthetic Risk Indicator
(The degree of deviation from the benchmark)

Three qualitative management classes

- Limited
- Intermediate
- Considerable

2nd Pillar
Synthetic risk indicator

Synthetic Risk Indicator
(The degree of deviation from the benchmark)

Time evolution of the volatility

Mapping of the qualitative risk classes into corresponding volatility intervals

Time evolution of the "intensity" of the management style

Mapping of each management class into corresponding intervals of a suitable measure

2nd Pillar: benchmark products

2nd Pillar
Synthetic risk indicator

Mapping of each management class into corresponding intervals of a suitable measure

Choice of a proper Volatility Measure:
the *delta-vol*
$\Delta\sigma = \sigma_F - \sigma_B$

Risk Classes	Delta-Vol Intervals					
	Limited		Intermediate		Considerable	
	$\Delta\sigma_{min}$	$\Delta\sigma_{max}$	$\Delta\sigma_{min}$	$\Delta\sigma_{max}$	$\Delta\sigma_{min}$	$\Delta\sigma_{max}$
Low	-0.118%	0.118%	-0.176%	0.176%	-0.235%	0.235%
Medium-Low	-0.239%	0.239%	-0.358%	0.358%	-0.477%	0.477%
Medium	-0.600%	0.600%	-0.900%	0.900%	-1.200%	1.200%
Medium-High	-1.250%	1.250%	-1.875%	1.875%	-2.500%	2.500%
High	-3.125%	3.125%	-4.668%	4.668%	-6.249%	6.249%
Very High	-6.250%	6.250%	-9.375%	9.375%	-12.500%	12.500%

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    - unbundling
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    - characterization of the necessary condition in the space of returns
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  - return target products

2nd Pillar: risk target and benchmark products

2nd Pillar
Synthetic risk indicator

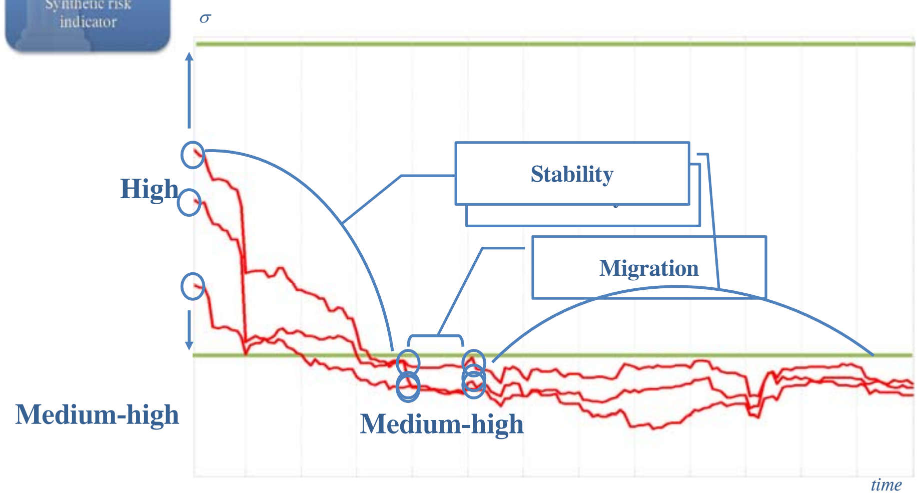
Migration of the Synthetic Risk Indicator

Migrations of the risk profile are persistent changes either of the degree of risk or of the degree of deviation from the benchmark which can significantly affect investors assessment of the non-equity product.

2nd Pillar: risk target and benchmark products

2nd Pillar
Synthetic risk indicator

Migration of the Synthetic Risk Indicator
(degree of risk)



2nd Pillar
Synthetic risk indicator

Migration of the Synthetic Risk Indicator

In order to correctly detect migrations, the width of both volatility and *delta-vol* intervals must be adequately set with respect to the period taken as a reference to assess the occurrence of these phenomena.

Too wide intervals could result in an artificial reduction in the number of migrations detected.

Too narrow intervals could result in an excessive number of migrations, many of them being spurious.

2nd Pillar
Synthetic risk indicator

Migration Rule
(degree of risk)

the iterative procedure guarantees that a product belonging to a given risk class does not breach the GARCH adaptive band more than 5% of the days in 1 year

↕

no more than 16 days over 250

2nd Pillar
Synthetic risk indicator

Migration Rule
(degree of risk)

BUT

migration risk is measured against fixed volatility intervals

hence

output intervals are inherently prudential w.r.t. the 3 months migration rule

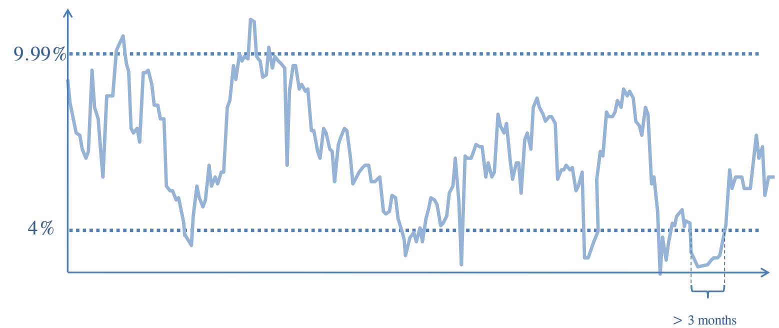
hence

output intervals are wide enough to avoid spurious migrations…

2nd Pillar
Synthetic risk indicator

Migration Rule
(degree of risk)

as confirmed by back-testing simulations:

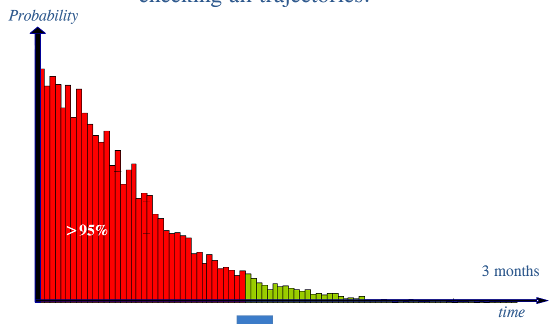


only 1 outlier lasting more than 3 months

2nd Pillar
Synthetic risk indicator

Migration Rule
(degree of risk)

checking all trajectories:



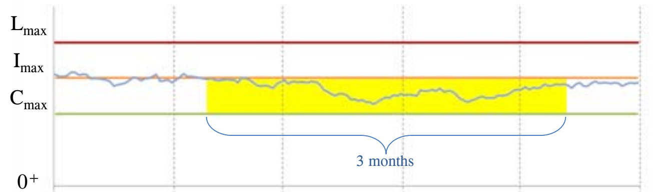
rolling daily check over the last 3 months

2nd Pillar
Synthetic risk indicator

Migration Rule
(degree of deviation from the benchmark)

Case A/B: no migration

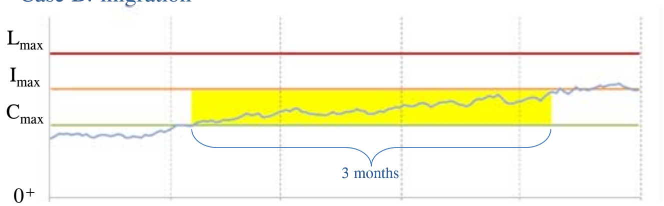
Case A: migration



2nd Pillar
Synthetic risk indicator

Migration Rule
(degree of deviation from the benchmark)

Case B: migration



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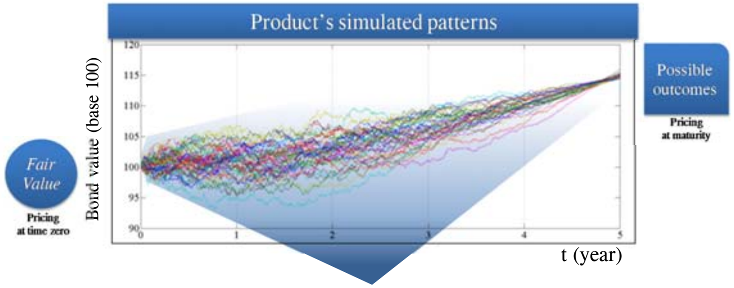
    - o characterization of the necessary condition in the space of returns
    - o how to determine a consistent series of Time Horizons
  - ➢ return target products

2nd Pillar: return target products

2nd Pillar
Synthetic risk indicator

"Risk target" products | "Benchmark" products | "Return target" products

In "return target" products the analysis of the volatility measures implicit in the probability distribution of the potential returns makes it possible to determine the risk class



DEGREE OF RISK

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    - first passage time
    - mimimun Recommended Time Horizon
    - characterization of the necessary condition in the space of returns
    - how to determine a consistent series of Time Horizons
  - return target products

3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

The Recommended Investment Time Horizon

Investment time horizon consistent with the risk-return profile and the costs associated with the product.

3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

"Risk target" products | "Benchmark" products | "Return target" products

The recommended investment time horizon

…for "risk-target" and benchmark products, the recommended investment time horizon is calculated as the *break-even* time, i.e. the minimum time required to recover initial costs and to off-set running costs, *at least once*, from a probabilistic point of view.

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3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

The recommended investment time horizon

In analytical terms, the probability of the event:

The investment recovers the initial costs and to off-sets the running costs at least once

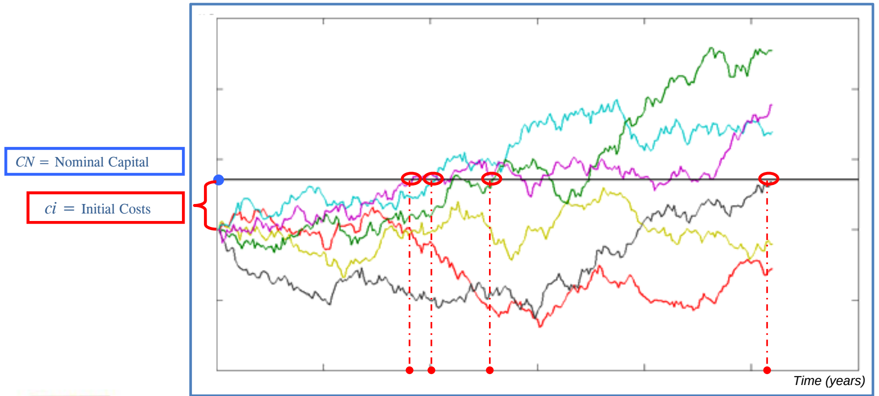
can be calculated through the concept of

First Passage Time

3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

First Passage Time:
First time (expressed in years) such that the value of the Invested Capital (CI) recovers the initial costs and off-sets the running costs.



3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

First Passage Time:

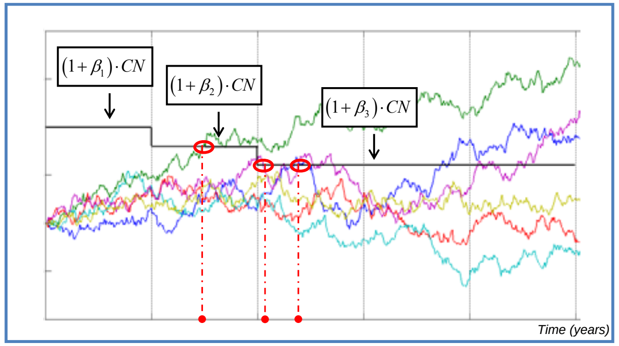
The costs treshold, depending from the presence of redemption's costs, can be variable

Deterministic Barrier	Stochastic Barrier
Fixed redemption's costs	Variable redemption's costs

3rd Pillar
The recommended investment horizon

Redemption's costs in percentage β_k of the Nominal Capital where β_k takes β_1 β_2 β_3, …, β_n values for different time intervals

$$CN = CI_t - \beta_k \cdot CN$$
$$\Updownarrow$$
$$CN + \beta_k \cdot CN = CI_t$$
$$\Updownarrow$$
$$CI_t = (1+\beta_k) \cdot CN$$



3rd Pillar
The recommended investment horizon

First Passage Time:

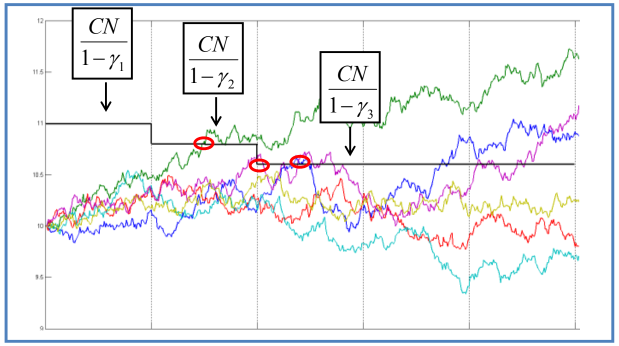
The costs threshold, depending from the presence of redemption's costs, can be variable

Deterministic Barrier	Stochastic Barrier
Fixed redemption's costs	Variable redemption's costs

3rd Pillar
The recommended investment horizon

Redemption's costs in percentage γ_k of the Invested Capital, where γ_k is variable with respect to time

$$CN = CI_t - \gamma_k \cdot CI_t$$
$$\Updownarrow$$
$$CN + \gamma_k \cdot CI_t = CI_t$$
$$\Updownarrow$$
$$CI_t = \frac{CN}{1-\gamma_k}$$



3rd Pillar
The recommended investment horizon

The probability of the event:

The investment recovers the initial costs and off-sets the running costs at least once

given a confidence level α, uniquely identifies a time T^* on the cumulative distribution function of the first passage times, i.e.:

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \le T\right] = \alpha \right\}$$

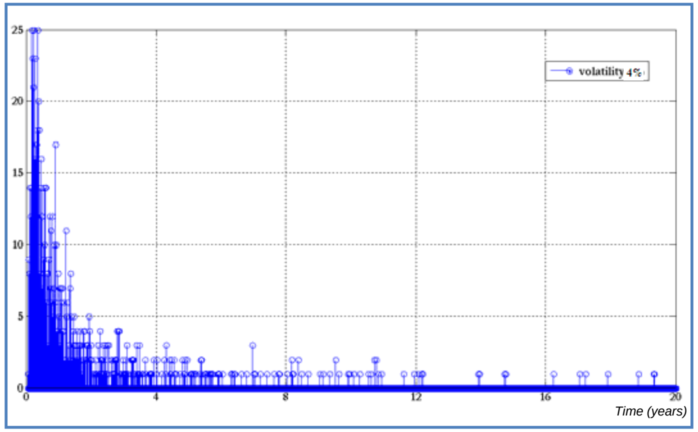
where

$$t^* = \inf\left[t \in \Re^+ : CI_t > CN\right]$$

is the first passage time

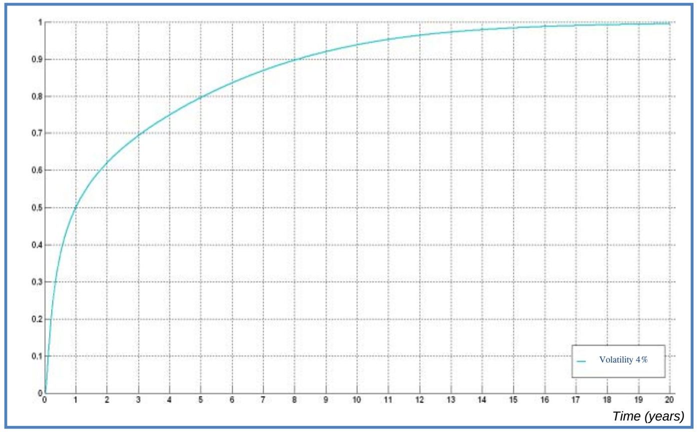
3rd Pillar
The recommended investment horizon

1. Calculation of the probability distribution of the first passage times:



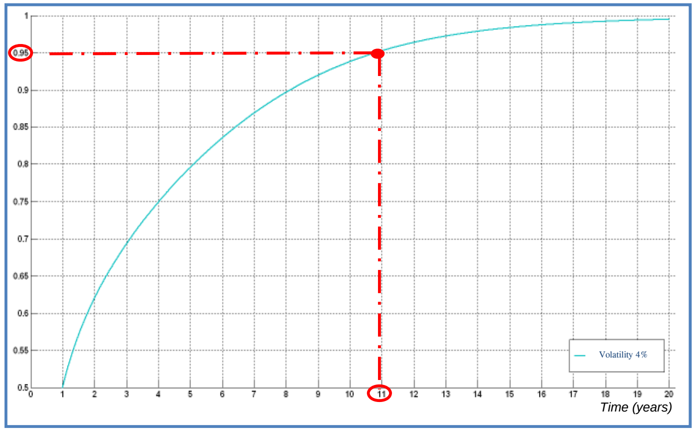
3rd Pillar
The recommended investment horizon

2. Derivation of the cumulative distribution function of the first passage times:



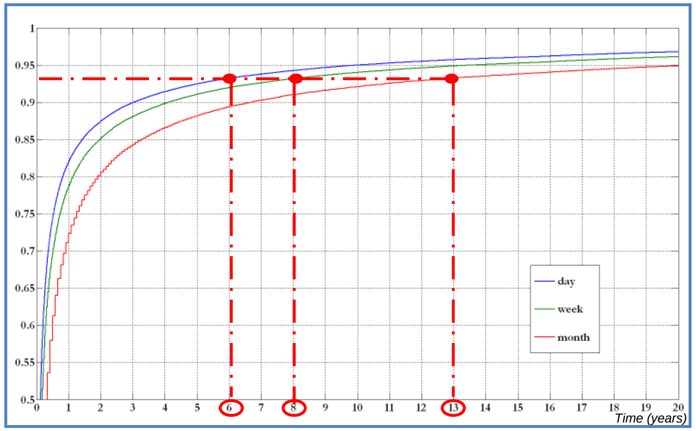
3rd Pillar
The recommended investment horizon

3. The confidence level α uniquely identifies T^* on the cumulative distribution function of the first passage times:



3rd Pillar
The recommended investment horizon

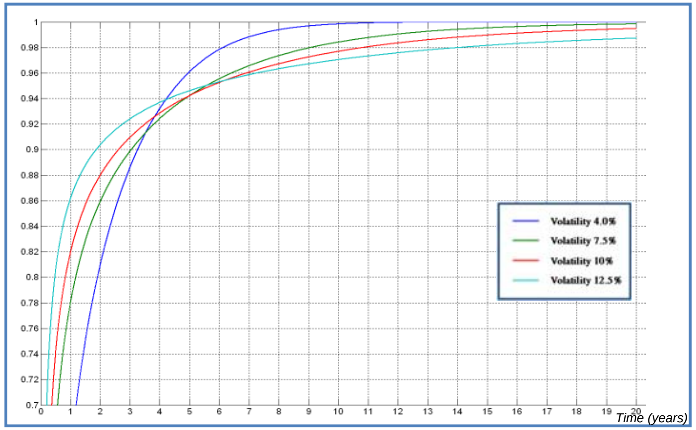
3. The discretization step is relevant in the determination of the cumulative probability function, conditioning the identification of the time horizon, given a fixed level of confidence:



3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

When many probability distribution functions are considered, letting varying volatilities and costs, the problem of correctly identifying a set of minumum thresholds arises:



3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

Anyway, the recommended minimum investment time horizon…

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \leq T\right] = \alpha \right\}$$

…. Must be coherent with the principle

+ VOLATILITY' + TIME HORIZON

3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

Anyway, the recommended minimum investment time horizon…

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \leq T\right] = \alpha \right\}$$

…. Must be coherent with the principle

+ VOLATILITY' + TIME HORIZON

The correct way to solve the problem is to set up an operative procedure to select properly each treshold according to the above principle

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Connection between probability, volatility and costs

First passage times for the break-even barrier are monitored at infinitesimal time intervals:

$dt \to 0$

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \le T\right] = \alpha \right\}$$

$$\mathrm{P}\left[t^* \le T\right] = N\left(d_2\left(\frac{CI_0}{CN}\right)\right) + \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \cdot N\left(-d_2\left(\frac{CN}{CI_0}\right)\right)$$

$$d_2(x) = \frac{\log x + \left(\bar{r} - cr - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

Connection between probability, volatility and costs

Asymptotic properties: $T \to \infty$

cr : recurrent costs as a fixed %

$dt \to 0$

$$\lim_{T\to\infty} \mathrm{P}\left[t^* \le T\right] = \begin{cases} 1 & \text{if} \quad (\bar{r} - cr) \ge \frac{1}{2}\sigma^2 \\ \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} & \text{if} \quad (\bar{r} - cr) < \frac{1}{2}\sigma^2 \end{cases}$$

Connection between probability, volatility and costs

Under our assumptions:

$dt \to 0$

$$\lim_{T\to\infty} \mathrm{P}\left[t^* \le T\right] = \begin{cases} 1 & \text{if} \quad (\bar{r} - cr) \ge \frac{1}{2}\sigma^2 \\ \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} & \text{if} \quad (\bar{r} - cr) < \frac{1}{2}\sigma^2 \end{cases}$$

For a given level of costs, it is possible to analytically derive the connection between volatility and time horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

FIRST ORDER SENSITIVITY ANALYSIS

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1}\right)$$

FIRST ORDER ASYMPTOTIC CONDITION

3rd Pillar – The recommended investment horizon

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

$$\frac{dP}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \right)$$

1. $(\bar{r}-cr) > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$
2. $(\bar{r}-cr) \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$

The existence of two alternative states of nature requires to verify whether both of them make sense in financial terms under the risk-neutral measure.

3rd Pillar – The recommended investment horizon

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

$$\frac{dP}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1} \right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$
2. $\bar{r} \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$

Being running costs a specific feature of any financial product they would interfere with the task of idenfying which of the two conditions has a sound financial meaning. Therefore, they will be temporarily neglected.

3rd Pillar – The recommended investment horizon

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

$$\frac{dP}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1} \right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$
2. ~~$\bar{r} \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$~~

Since it is safe to assume a positive interest rate *r* in financial markets, only condition 1. correctly captures the connection between volatility and time horizon.

3rd Pillar – The recommended investment horizon

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

$$\frac{dP}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1} \right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$
2. ~~$\bar{r} \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$~~

As $T \to \infty$ condition 1. implies that the cumulative distribution function P is a strictly decreasing function of the volatility, i.e.:

$$\forall \sigma_i, \sigma_j \in \Re^+, \sigma_j > \sigma_i \Rightarrow P(\sigma_j) < P(\sigma_i)$$

3rd Pillar
The recommended investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{dP}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1} \right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$
2. ~~$\bar{r} \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$~~

In other words, for a given a confidence level, as the volatility grows, the recommended investement time horizon increases as well:

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

3rd Pillar
The recommended investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{dP}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1} \right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$
2. $\bar{r} \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$

$\exists T^* \in [0, \infty[: \frac{dP}{d\sigma} = 0$

Furthermore, condition 1. alone is sufficient to guarantee a minimum time T^* beyond which the following strong condition holds:

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

3rd Pillar
The recommended investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{dP}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \right)$$

$cr > 0$

1. $(\bar{r}-cr) > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$
2. ~~$(\bar{r}-cr) \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$~~

$\exists T^* \in [0, \infty[: \frac{dP}{d\sigma} = 0$

Generalizing…

3rd Pillar
The recommended investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d^2P}{d\sigma^2} = \frac{4}{\sigma^4}(\bar{r}-cr)\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \cdot \left[1 + \frac{4(\bar{r}-cr)}{\sigma^2}\ln\left(\frac{CN}{CI_0}\right)\right]$$

$(\bar{r}-cr) > 0 \Rightarrow \frac{d^2P}{d\sigma^2} > 0$

SECOND ORDER ASYMPTOTIC CONDITION

Second Order Sensitivity Analysis

3rd Pillar
The recommended investment horizon

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

1.
$$\begin{cases} (\bar{r} - cr) > 0 \Leftrightarrow \dfrac{dP}{d\sigma} < 0 \\ (\bar{r} - cr) > 0 \Rightarrow \dfrac{d^2P}{d\sigma^2} > 0 \end{cases}$$

$$\exists T^* \in [0, \infty[: \frac{dP}{d\sigma} = 0$$

2. ~~$(\bar{r} - cr) \leq 0 \Leftrightarrow \dfrac{dP}{d\sigma} \geq 0$~~

Summarizing the results of the asymptotic analisys in continuous time:

- As T →∞, for given a confidence level, more volatility implies a larger recommended investment time horizon
- It is always possible to find a <u>minimum</u> and finite time T^*, beyond which the strong condition

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

holds

3rd Pillar
The recommended investment horizon

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

1.
$$\begin{cases} (\bar{r} - cr) > 0 \Leftrightarrow \dfrac{dP}{d\sigma} < 0 \\ (\bar{r} - cr) > 0 \Rightarrow \dfrac{d^2P}{d\sigma^2} > 0 \end{cases}$$

2. ~~$(\bar{r} - cr) \leq 0 \Leftrightarrow \dfrac{dP}{d\sigma} \geq 0$~~ ⇒ $\bar{T} = x \ years$

It is necessary to drop from the analysis those cases which yield condition 2 (i.e. whenever the drift positiveness is not satisfied). Under such a condition, the recommended time horizon is set *by default* equal to a pre-defined limit *x*.

3rd Pillar
The recommended investment horizon

<u>DETERMINATION OF THE INVESTIMENT TIME HORIZON</u>

General Framework:

$$\begin{cases} T \to \infty \\ dt \to 0 \\ P(\infty, \sigma) \\ 1. \begin{cases} (\bar{r} - cr) > 0 \Leftrightarrow \dfrac{dP}{d\sigma} < 0 \\ (\bar{r} - cr) > 0 \Rightarrow \dfrac{d^2P}{d\sigma^2} > 0 \end{cases} \end{cases} \Rightarrow \begin{cases} T \ \ finite \\ dt \to 0 \\ P(T, \sigma) \\ (\bar{r} - cr) > 0 \Leftrightarrow \lim_{T \to \infty} \dfrac{\partial P(T, \sigma)}{\partial \sigma} < 0 \\ (\bar{r} - cr) > 0 \Rightarrow \lim_{T \to \infty} \dfrac{\partial^2 P(T, \sigma)}{\partial \sigma^2} > 0 \end{cases}$$

In order to determine effectively the investment time horizon, it is necessary to abandon the asymptotic environment and to shift the analysis of condition 1. in a finite time's framework.

3rd Pillar
The recommended investment horizon

<u>DETERMINATION OF THE INVESTIMENT TIME HORIZON</u>

FIRST ORDER SENSITIVITY ANALYSIS

$$\frac{\partial P(T, \sigma)}{\partial \sigma}$$

At a finite time T, the asymptotic relationship $\lim_{T \to \infty} \dfrac{\partial P(T, \sigma)}{\partial \sigma} < 0$

determines the existence and the unicity of a time:

$$T^*_{\bar{\sigma}} = \inf \left\{ T \in [0, \infty[\ : \ \left. \frac{\partial P(T, \sigma)}{\partial \sigma} \right|_{\sigma = \bar{\sigma}} < 0 \right\}$$

such that:

$$(\bar{r} - cr) > 0 \Rightarrow \begin{cases} \left. \dfrac{\partial P(T, \sigma)}{\partial \sigma} \right|_{\sigma = \bar{\sigma}} > 0 & if \quad 0 \leq T < T^*_{\bar{\sigma}} \\ \left. \dfrac{\partial P(T, \sigma)}{\partial \sigma} \right|_{\sigma = \bar{\sigma}} \leq 0 & if \quad T \geq T^*_{\bar{\sigma}} \end{cases}$$

3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

FIRST ORDER SENSITIVITY ANALYSIS

$$\frac{\partial P(T,\sigma)}{\partial \sigma}$$

At a finite time T, the sufficient condition of the first order that allows to state the core relationship

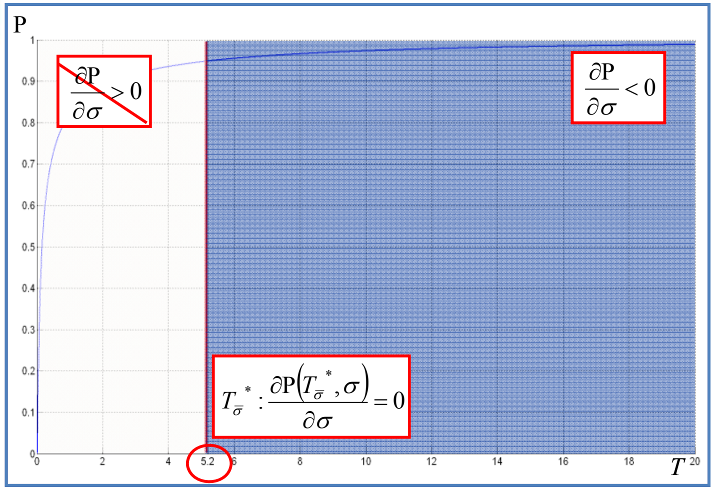
+ volatility + time horizon

is then specified in the following form:

$$\begin{cases} \left.\frac{\partial P(T,\sigma)}{\partial \sigma}\right|_{\sigma=\bar{\sigma}} > 0 & if \quad 0 \leq T < T_{\bar{\sigma}}^{*} \\ \left.\frac{\partial P(T,\sigma)}{\partial \sigma}\right|_{\sigma=\bar{\sigma}} \leq 0 & if \quad T \geq T_{\bar{\sigma}}^{*} \end{cases}$$

3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON



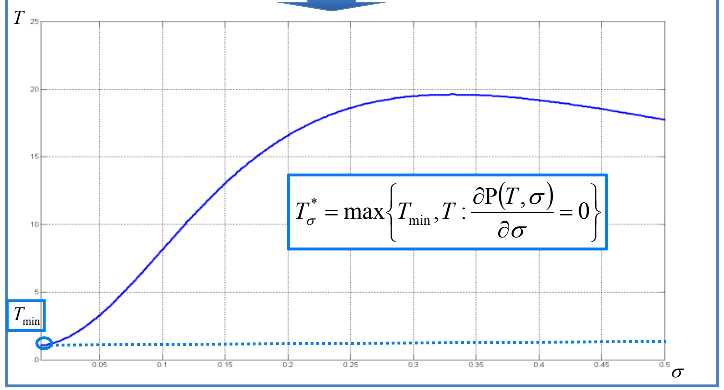
3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

FIRST ORDER SENSITIVITY ANALYSIS

$$\frac{\partial P(T,\sigma)}{\partial \sigma}$$

Letting σ vary, the function of minimum times T_{σ}^{*} is built

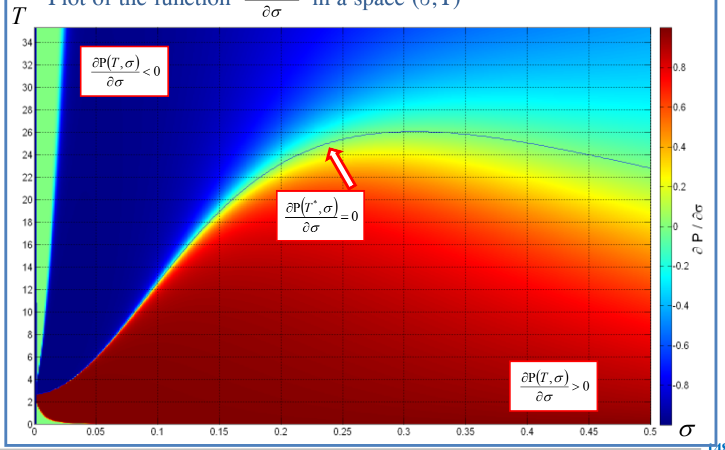


3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

FIRST ORDER SENSITIVITY ANALYSIS

Plot of the function $\frac{\partial P(T,\sigma)}{\partial \sigma}$ in a space (σ,T)



3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS

$$\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}$$

The sign of the quantity:

$$\left.\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}\right|_{T=T_\sigma^*}$$

determines the behaviour of the function of minimum times, i.e.:

$$\left.\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}\right|_{T=T_\sigma^*} > 0 \Rightarrow T_\sigma^* \text{ increasing}$$

$$\left.\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}\right|_{T=T_\sigma^*} < 0 \Rightarrow T_\sigma^* \text{ decreasing}$$

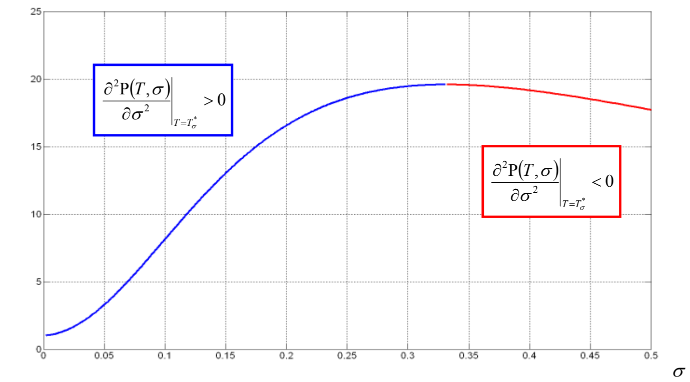
3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS

$$\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}$$

Plot of the function T_σ^*



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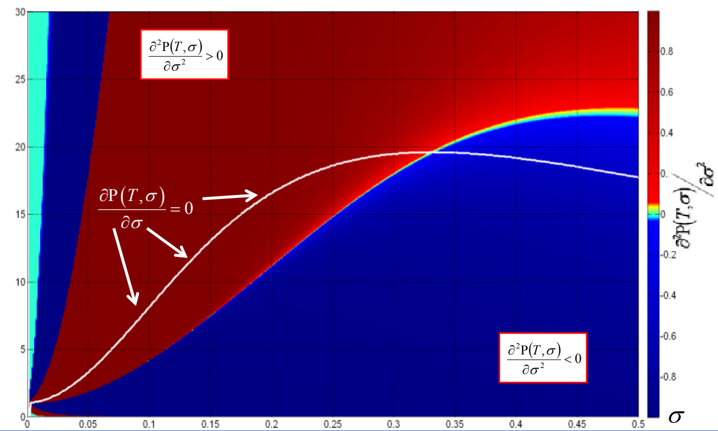
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

Plot of the function $\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}$ in a space (σ,T)



3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS

$$\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}$$

Given the monotonicity condition of the probability distribution with respect to volatility, i.e.:

$$\forall \sigma_i, \sigma_j \in \Re^+, \sigma_j > \sigma_i \Rightarrow P(\infty,\sigma_j) < P(\infty,\sigma_i)$$

In order to fulfill this condition, it's necessary to restrict the analysis in the region where the probability function is strictly increasing, i.e.:

$$\left.\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}\right|_{T=T_\sigma^*} > 0 \Rightarrow T_\sigma^* \text{ increasing}$$

~~$\left.\frac{\partial^2 P(T,\sigma)}{\partial \sigma^2}\right|_{T=T_\sigma^*} < 0 \Rightarrow T_\sigma^* \text{ decreasing}$~~

3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS

$$\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$$

Having defined the maximum time in the form:

$$\begin{cases} \sigma \in \Re^+ \\ T_{max} \in T^*_\sigma \end{cases} : \frac{\partial^2 P(T_{max},\sigma)}{\partial\sigma^2} = 0$$

The sufficient condition of the 2° order is specified as:

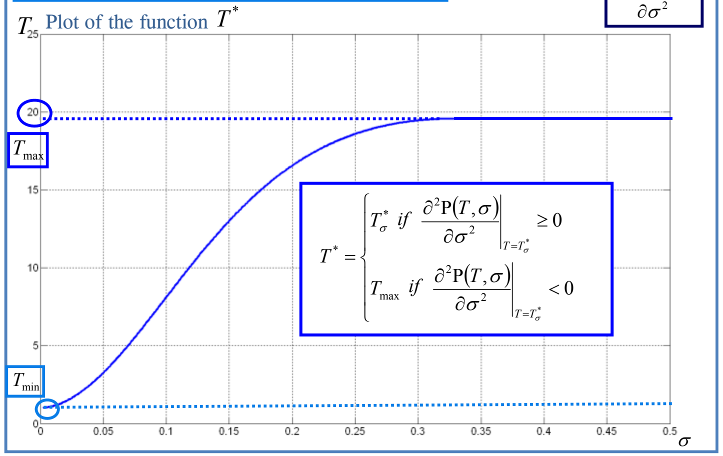
$$T^* = \begin{cases} T^*_\sigma \;\; se \;\; \left.\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}\right|_{T=T^*_\sigma} \geq 0 \\ T_{max} \;\; se \;\; \left.\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}\right|_{T=T^*_\sigma} < 0 \end{cases}$$

3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS

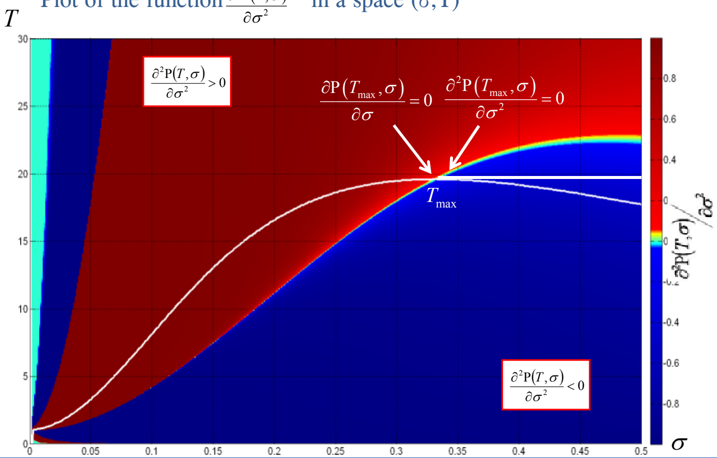
$$\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$$



3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

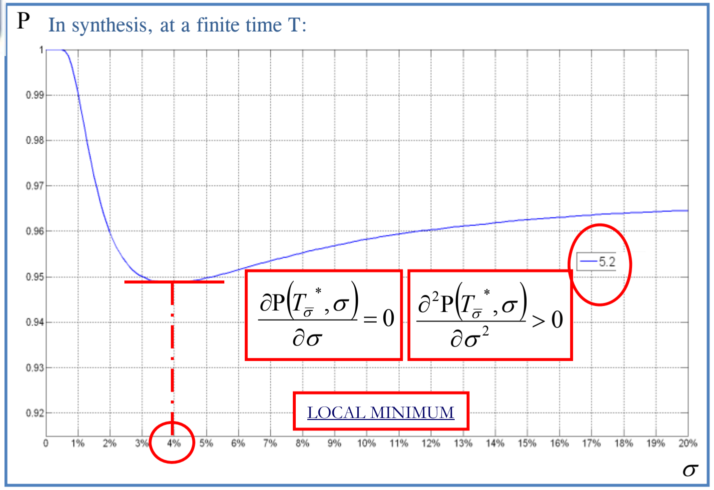
Plot of the function $\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$ in a space (σ,T)



3rd Pillar
The recommended investment horizon

DETERMINATION OF THE INVESTIMENT TIME HORIZON

In synthesis, at a finite time T:



3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

When the methodology is implemented in more general frameworks where rates and volatilities are variable, the closed formula approach has to be abandoned and Monte Carlo simulations are required to proceed in the analysis.

In the following the determination of the minimum time horizon is specified in a discrete setting characterized by an increasing sequence of volatilities and a given costs class

3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

STRONG CONVERGENCE LEMMA for times

Given a sequence of financial products F_j with volatility σ_j and recalling the first order sufficient condition:

$$T_\sigma^* = \max\left\{T_{\min}, T : \frac{\partial \mathrm{P}(T,\sigma)}{\partial \sigma} = 0\right\}, \quad \forall \sigma \in \Re^+$$

the first order sufficient condition can be specified for the class of products F_j in the following form:

$$T_{\sigma_j}^{\varepsilon_j} : \mathrm{P}\left(T_{\sigma_j}^{\varepsilon_j}, \sigma_{j+1}\right) = \mathrm{P}\left(T_{\sigma_j}^{\varepsilon_j}, \sigma_j\right)$$

It therefore holds the following strong convergence relation with respect to times:

$$\lim_{\sigma_{j+1} \to \sigma_j} T_{\sigma_j}^{\varepsilon_j} = T_{\sigma_j}^*$$

where $\varepsilon_j = (\sigma_{j+1} - \sigma_j) > 0.$

3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In order to have an intuitive explanation of the lemma, let's consider the following volatility levels:

$\sigma - \varepsilon, \varepsilon \in \Re^+$ | σ | $\sigma + \varepsilon, \varepsilon \in \Re^+$

and the respective probability distribution functions, i.e.:

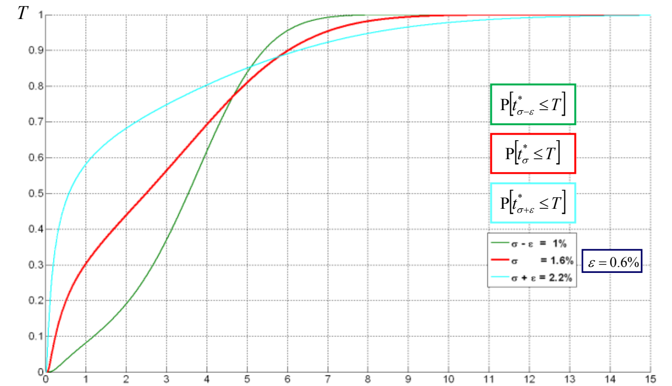
$\mathrm{P}[t_{\sigma-\varepsilon}^* \le T]$ | $\mathrm{P}[t_\sigma^* \le T]$ | $\mathrm{P}[t_{\sigma+\varepsilon}^* \le T]$

3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



3rd Pillar: recommended investment time horizon

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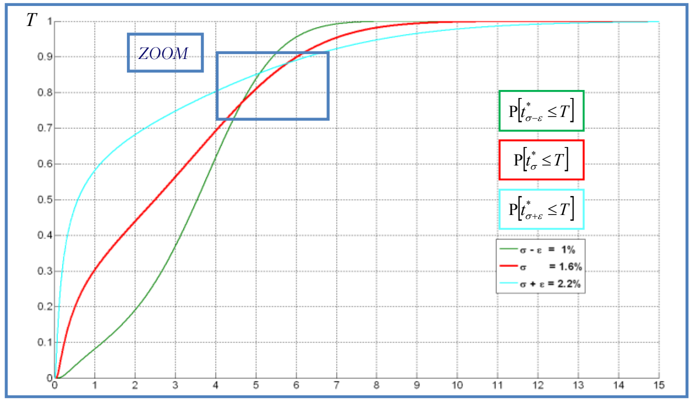
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



3rd Pillar: recommended investment time horizon

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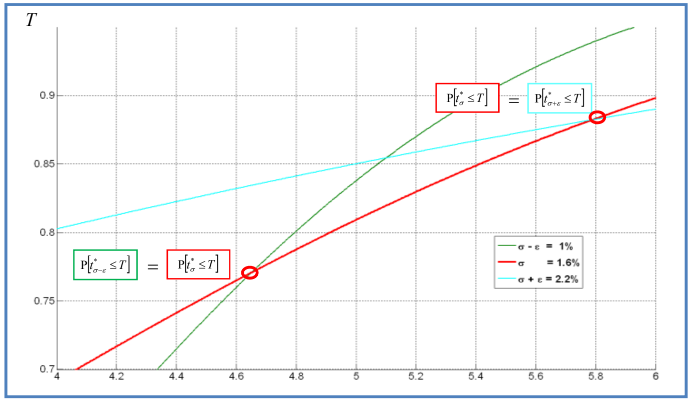
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

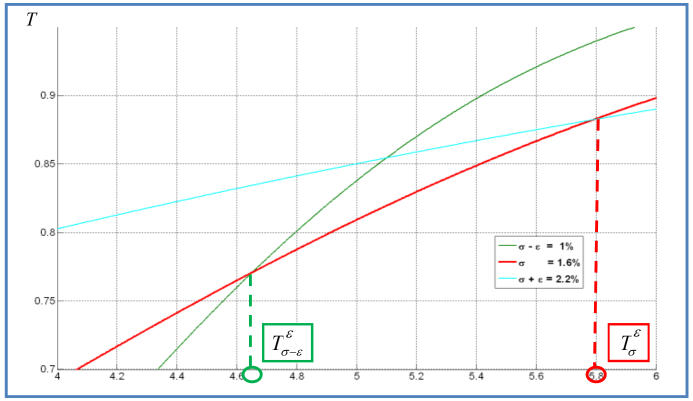


3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



3rd Pillar: recommended investment time horizon

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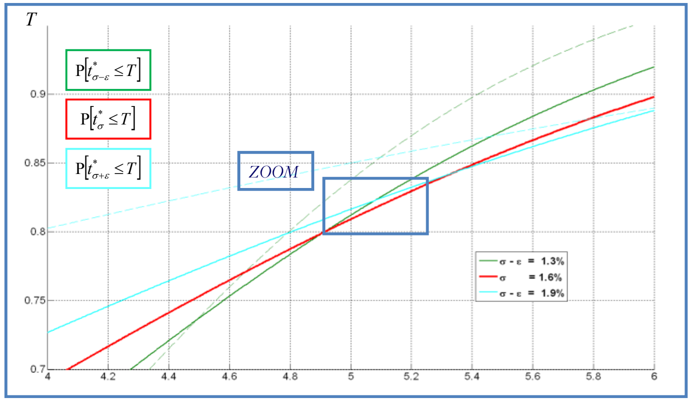
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



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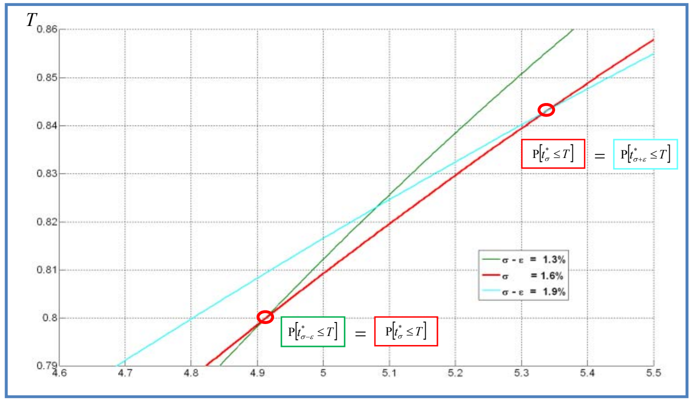
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



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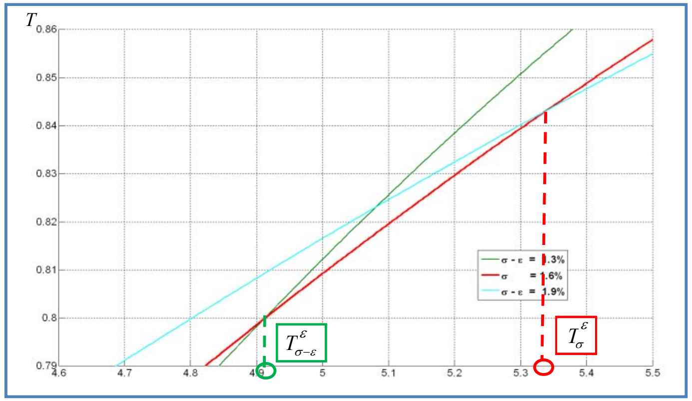
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

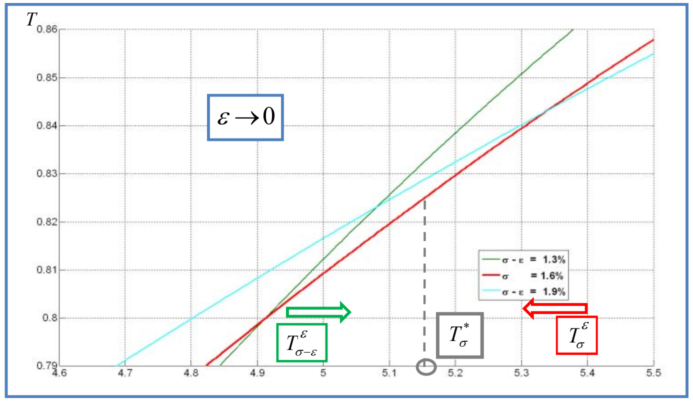
FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

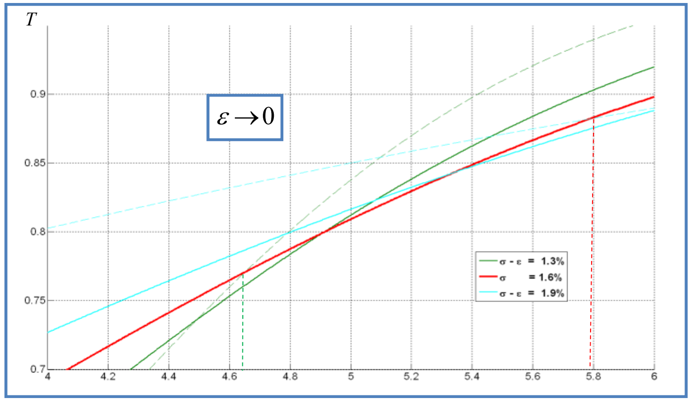


3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In synthesis..



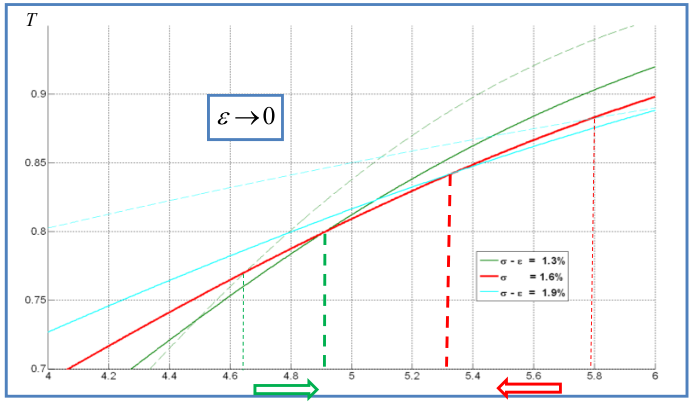
3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In synthesis..



3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

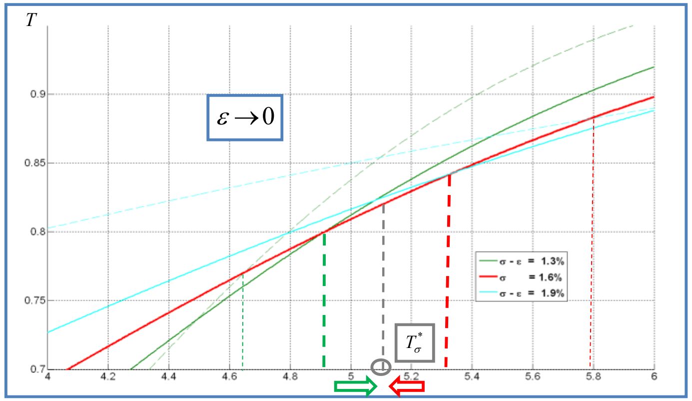
+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In synthesis..

$\sigma = 1.6\%$

$T^*_\sigma = 5.1$ years



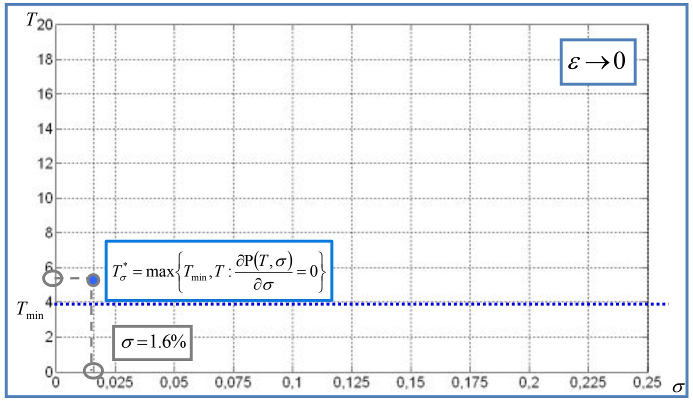
3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

The time is characterized on the curve of minimum times..



3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

Generalizing the lemma for all σ, the following characterization of the first order sufficient condition is given:

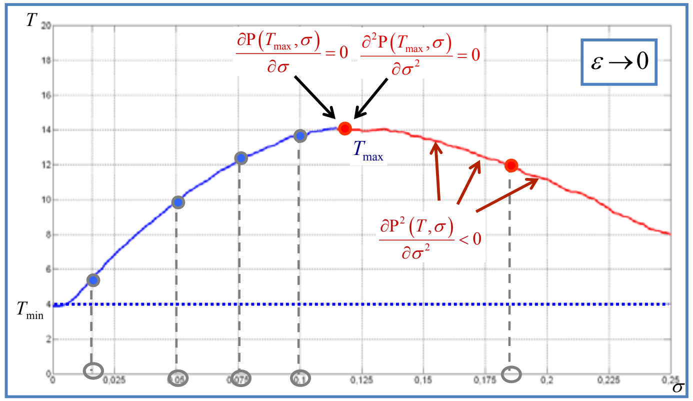
$\sigma-\varepsilon, \varepsilon \in \Re^+$	σ	$\sigma+\varepsilon, \varepsilon \in \Re^+$
$P\left[t^*_{\sigma-\varepsilon} \le T\right]$	$P\left[t^*_{\sigma} \le T\right]$	$P\left[t^*_{\sigma+\varepsilon} \le T\right]$

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

The time is characterized on the curve of minimum times..



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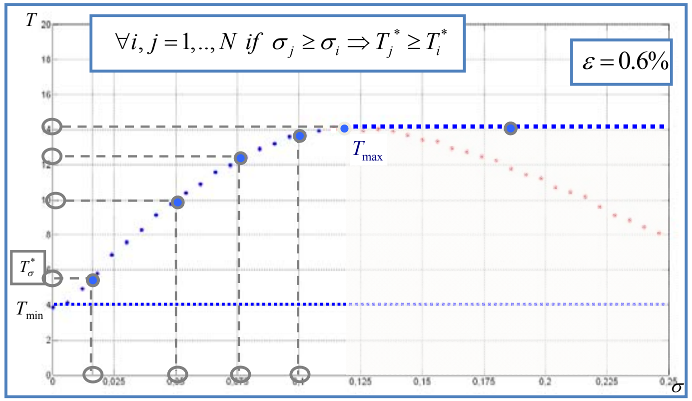
SECOND ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

Weak monotonicity condition of times w.r.t. volatility



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FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

Formally, for any sequence of products with volatility σ_j, defined in a given class of costs (ci,cr):

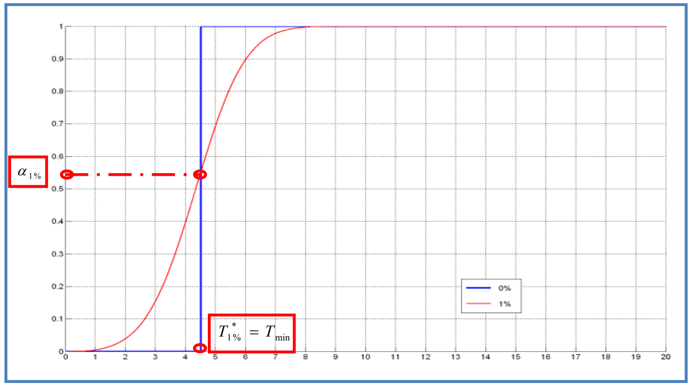
Strong convergence lemma for times First order sufficient condition	Weak monotonicity condition of times w.r.t. volatility Second order sufficient condition

$$\forall j=1,\dots N,\sigma_{j+1}>\sigma_j,$$
$$T_{j+1}^*=\max\left\{T_j^*,\ T\in[T_{min},T_{max}]:\mathrm{P}\left[t_{\sigma_{j+1}}^*\leq T\right]=\mathrm{P}\left[t_{\sigma_j}^*\leq T\right]=\alpha_{j+1}^*\right\}$$

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

Practical Method to derive a sequence of time horizons

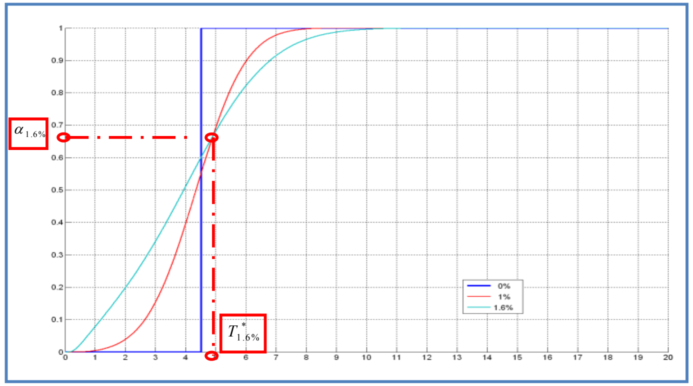


3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

Practical Method to derive a sequence of time horizons



3rd Pillar: recommended investment time horizon

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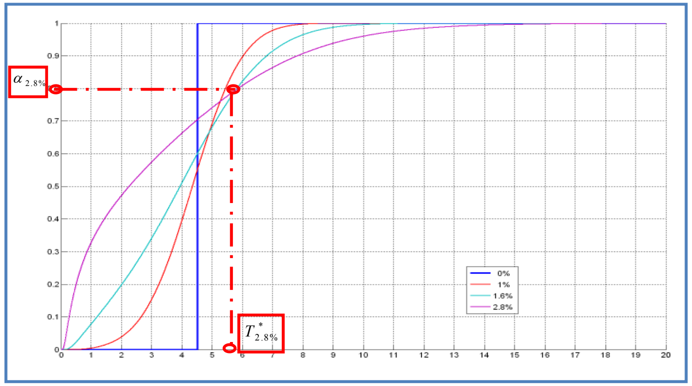
Practical Method to derive a sequence of time horizons



3rd Pillar: recommended investment time horizon

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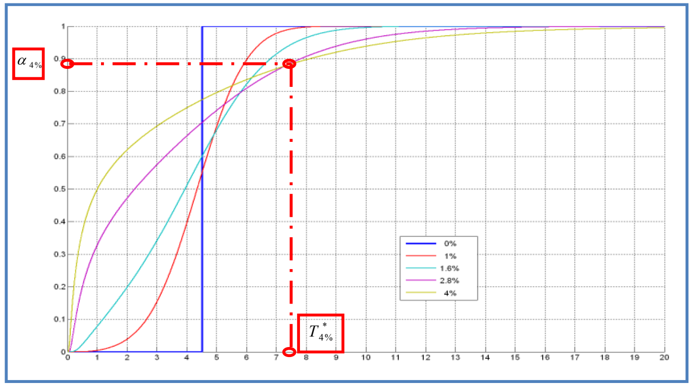
Practical Method to derive a sequence of time horizons



3rd Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

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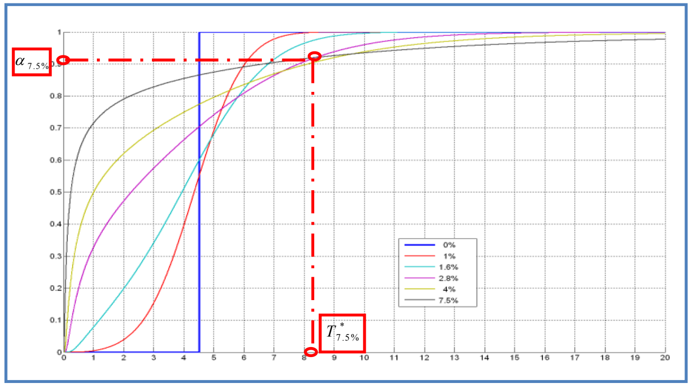
Practical Method to derive a sequence of time horizons



3^{rd} Pillar: recommended investment time horizon

3rd Pillar
The recommended investment horizon

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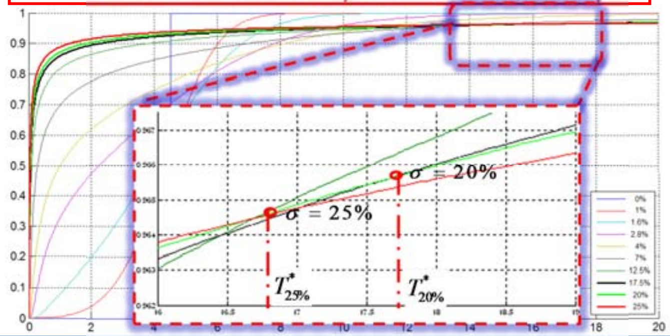
Practical Method to derive a sequence of time horizons

Violation of the Monotonicity Condition w.r.t. volatility



3^{rd} Pillar: recommended investment time horizon

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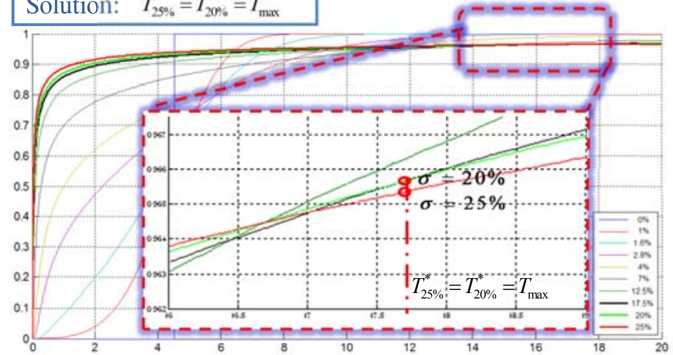
Practical Method to derive a sequence of time horizons

Solution: $T^*_{25\%} = T^*_{20\%} = T_{max}$



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Risk Based Approach towards Transparency on Non-Equity Investment Products

Marcello Minenna – Head of Quantitative Analysis Unit, Consob

Columbia University — Department of Mathematics, Columbia University in the City of New York