CONSOB
Commissione Nazionale per le Società e la Borsa

Consumer Protection via quantitative risk disclosure

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CHIEF **RISK OFFICERS' CLUB – REFLECT, REFORM, REBUILD**
16-17 November 2010

Syllabus

Preliminaries

Three-pillars approach:

1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

Preliminaries

Non-equity Investment products should be classified according to their financial characteristics and not to **"labels"** that are assigned by the issuer and/or by the European regulatory framework.

LABELS: Bond; Equity Mutual Funds; Structured ETF; Liquidity Mutual Funds; Bond ETF; Liquidity ETF; Equity Unit Linked; Structured Bond; Bond Unit Linked

RISK TARGET PRODUCT | BENCHMARK PRODUCT | RETURN TARGET PRODUCT

Preliminaries

Consob regulation on transparency on the risk profile of non-equity investment products is based on synthetic indicators – defined through the development of specific quantitative methods – in order to allow investors to take informed investment decisions.

Traditional narrative description of all possible risks associated with a predefined **"label"**

VS

Synthetic indicators robust, objective and backward verifiable

Bond; Equity Mutual Funds; Structured ETF; Liquidity Mutual Funds; Bond ETF; Liquidity ETF; Equity Unit Linked; Structured Bond; Bond Unit Linked

RISK TARGET PRODUCT | BENCHMARK PRODUCT | RETURN TARGET PRODUCT

Preliminaries

RETURNS | RISKS | INVESTMENT HORIZON

The key qualitative information is made objective by using a three-pillars approach.

1st Pillar	2nd Pillar	3rd Pillar
Unbundling and Probabilistic performance scenarios	Synthetic risk indicator	The recommended Investment horizon

Preliminaries

Investor decisions as a **sequential filtering problem**:

Time goal: liquidity/investment horizon
INVESTMENT HORIZON
(less than 3 years)

Bond; Equity Mutual Funds; Structured ETF; Liquidity Mutual Funds; Bond ETF; Liquidity ETF; Equity Unit Linked; Structured Bond; Bond Unit Linked

Risk profile: risk limit in terms of downside
RISKS
(medium-low)

Bond; Equity Unit Linked; Liquidity Mutual Funds; Bond ETF; Liquidity ETF; Structured Bond; Bond Unit Linked

Return goal: target returns
RETURNS
(maximum return)

Bond; Liquidity Mutual Funds; Liquidity ETF

Syllabus

Preliminaries

Three-pillars approach:

1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

1st Pillar: Unbundling and Probabilistic performance scenarios

RISK TARGET PRODUCT | BENCHMARK PRODUCT | RETURN TARGET PRODUCT

In "return target" products (e.g. corporate bonds) the connection between the pricing at time zero and the pricing at maturity is evident, as the probability table is a necessary step to obtain the unbundling of the price of the product at time 0.

Fair Value
Pricing at time zero

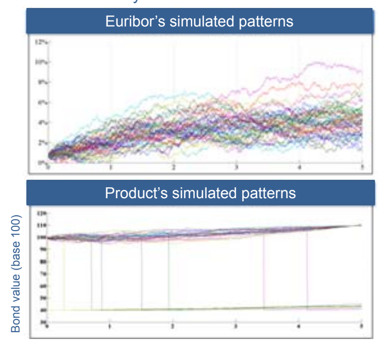
Unbundling

Returns probability distribution

Possible Outcomes
Pricing at maturity

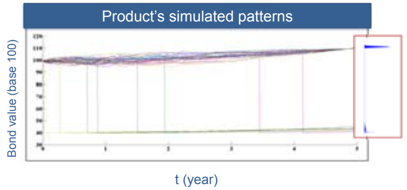
1st Pillar: Unbundling and Probabilistic performance scenarios

5 year fixed-rate bond



1st Pillar: Unbundling and Probabilistic performance scenarios

The final values of the bond at the end of the 5th year provide the probability distribution of potential returns (so-called *pricing* at maturity).



Possible Outcomes
Pricing at maturity

1st Pillar: Unbundling and Probabilistic performance scenarios

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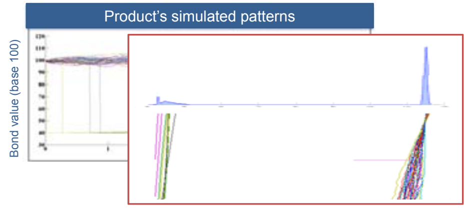
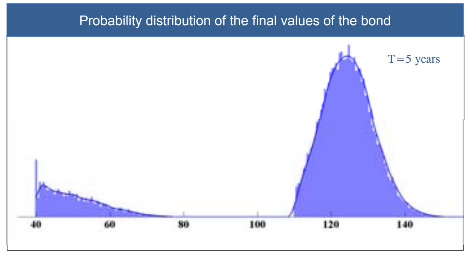


Possible Outcomes
Pricing at maturity

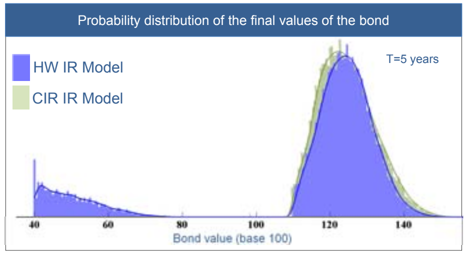
1st Pillar: Unbundling and Probabilistic performance scenarios

COMPLEXITY FOR RETAIL INVESTORS: The informative content of the entire probability distribution is very complex to handle for the average retail investor.



1st Pillar: Unbundling and Probabilistic performance scenarios

MODEL RISK: The shape of the probability distribution of potential returns is obviously dependent on the model's assumption.

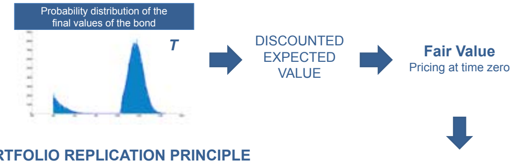


1st Pillar: Unbundling and Probabilistic performance scenarios

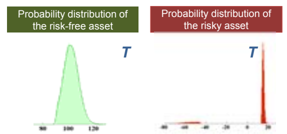
COMPLEXITY FOR RETAIL INVESTORS: STANDARD SOLUTION

1st Pillar: Unbundling and Probabilistic performance scenarios

COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (1)



PORTFOLIO REPLICATION PRINCIPLE



Theoretical value of the bond-like component
Risk-free asset

+

Theoretical value of the derivative component
Risky asset

1st Pillar: Unbundling and Probabilistic performance scenarios

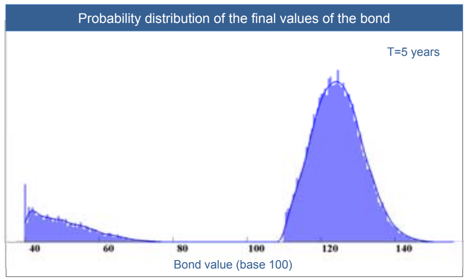
COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (1)

Financial investment table
(Unbundling)

A	Theoretical value of the bond-like component	…
B	Theoretical value of the derivative component	…
C = A + B	*Fair value*	…
D	Explicit costs	…
E	Implicit costs	…
F = C + D + E	Issue price	100

1st Pillar: Unbundling and Probabilistic performance scenarios

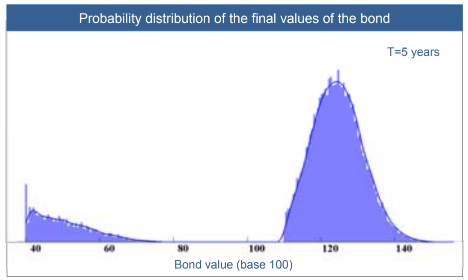
COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (2)



It's interesting to explore a different representation of the information contained in the probability distribution which could be useful for the average investor

1st Pillar: Unbundling and Probabilistic performance scenarios

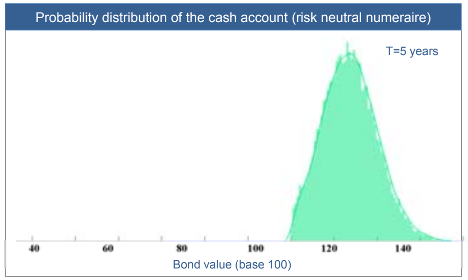
COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (2)



In order to provide the investor with a representation fair, easy to understand and resilient to the model's risk, a simple rescaling with respect to the risk-neutral measure numeraire is presented

1st Pillar: Unbundling and Probabilistic performance scenarios

COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (2)



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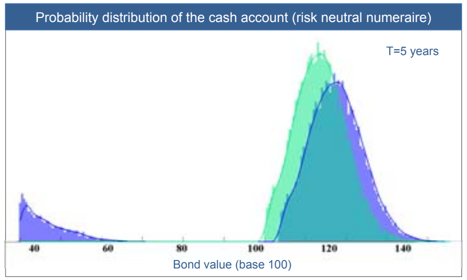
1st Pillar: Unbundling and Probabilistic performance scenarios

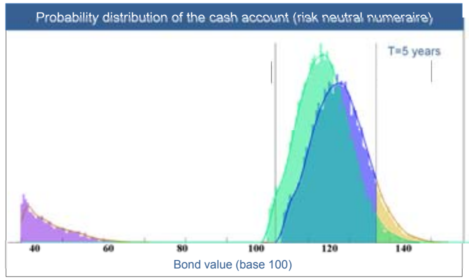
COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (2)



The superimposition of the product's probability distribution with the cash account naturally defines three different events which are effectively meaningful for the investor.

1st Pillar: Unbundling and Probabilistic performance scenarios

COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (2)



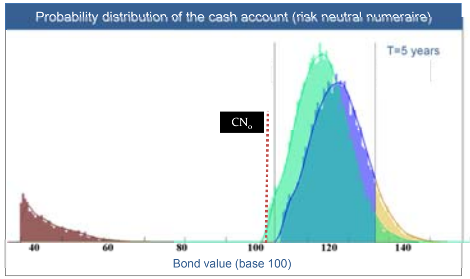
The performance is lower than the risk-free asset

The performance is in line with the risk-free asset

The performance is higher than the risk-free asset

1st Pillar: Unbundling and Probabilistic performance scenarios

COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (2)



The performance is negative

The performance is positive and lower than the risk-free asset

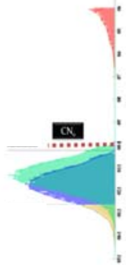
The performance is positive and in line with the risk-free asset

The performance is positive and higher than the risk-free asset

1st Pillar: Unbundling and Probabilistic performance scenarios

COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (2)

Probabilistic performance scenario table



SCENARIOS	PROBABILITY	MEDIAN VALUES
The performance is negative	%	€
The performance is positive but lower than the risk-free asset	%	€
The performance is positive and in line with the risk-free asset	%	€
The performance is positive and higher than the risk-free asset	%	€

1st Pillar: Unbundling and Probabilistic performance scenarios

COMPLEXITY FOR RETAIL INVESTORS: CONSOB REGULATION (1) e (2)

Connection between the pricing at time zero and the pricing at the end of recommended investment horizon

Time Zero — Financial investment table

A	Theoretical value of the bond-like component	…
B	Theoretical value of the derivative component	…
C = A + B	*Fair value*	…
D	Explicit costs	…
E	Implicit costs	…
F = C + D + E	Issue price	100

End of the recommended investment horizon — Table of probabilistic performance scenarios

SCENARIOS	PROBABILITY	MEDIAN VALUES
The performance is negative	%	€
The performance is positive but lower than the risk-free asset	%	€
The performance is positive and in line with the risk-free asset	%	€
The performance is positive and higher than the risk-free asset	%	€

1:1 Relationship

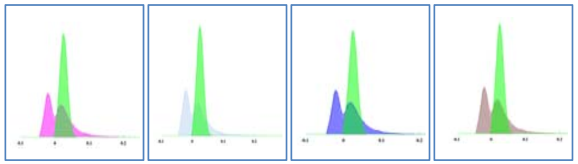
1st Pillar: Unbundling and Probabilistic performance scenarios

MODEL RISK: CONSOB REGULATION

The model risk arising from the right to freely use the proprietary models is solved with the reduction in granularity of events

Many possible choices…

Heston | Merton | V G | NIG



1st Pillar: Unbundling and Probabilistic performance scenarios

MODEL RISK: CONSOB REGULATION

The results of the various models show differences between each box of less than 5%

… the following output is obtained:

Heston

Scenarios	Probability	Median Values
The performance is negative	46.61 %	€ 90.50
The performance is positive but lower than the risk-free asset	3.39%	€ 101.26
The performance is positive and in line with the risk-free asset	33.28 %	€ 112.19
The performance is positive and higher than the risk-free asset	16.72 %	€ 139.93

Merton

Scenarios	Probability	Median Values
The performance is negative	42.695 %	€ 89.26
The performance is positive but lower than the risk-free asset	4.74%	€ 102.54
The performance is positive and in line with the risk-free asset	35.7%	€ 110.09
The performance is positive and higher than the risk-free asset	16.86 %	€ 142.65

V G

Scenarios	Probability	Median Values
The performance is negative	43.91 %	€ 91.25
The performance is positive but lower than the risk-free asset	5.23%	€ 102.1
The performance is positive and in line with the risk-free asset	36.8%	€ 109.24
The performance is positive and higher than the risk-free asset	14.06 %	€ 141.77

NIG

Scenarios	Probability	Median Values
The performance is negative	48.1%	€ 93.40
The performance is positive but lower than the risk-free asset	2.6%	€ 101.91
The performance is positive and in line with the risk-free asset	34.28 %	€ 114.23
The performance is positive and higher than the risk-free asset	15.02 %	€ 142.13

Syllabus

Preliminaries

Three-pillars approach:

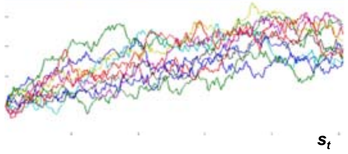
1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

2nd Pillar: Synthetic risk indicator

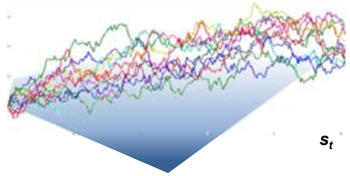
Volatility of the product's simulated returns



Volatility is the most immediate risk measure and
it has a one-to-one relationship with whatever loss measure
(VaR, ES, etc.)

2nd Pillar: Synthetic risk indicator

Volatility of the product's simulated returns



MEASUREMENT:
product's positioning inside
a grid of *n* volatility intervals

REPRESENTATION:
mapping of any volatility interval into
a corresponding qualitative risk class

Risk Classes	Volatility Intervals	
Very Low	$\sigma_{1,min}$	$\sigma_{1,max}$
Low	$\sigma_{2,min}$	$\sigma_{2,max}$
Medium-Low	$\sigma_{3,min}$	$\sigma_{3,max}$
Medium	$\sigma_{4,min}$	$\sigma_{4,max}$
Medium-High	$\sigma_{5,min}$	$\sigma_{5,max}$
High	$\sigma_{6,min}$	$\sigma_{6,max}$
Very High	$\sigma_{7,min}$	$\sigma_{7,max}$

2nd Pillar: Synthetic risk indicator

MEASUREMENT:
product's positioning inside
a grid of *n* volatility intervals

REPRESENTATION:
mapping of any volatility interval into
a corresponding qualitative risk class

Looking for the number of intervals
(so-called "n-tuple of risk classes")
allowing the best compromise between
investors' comprehension and
detail of the information conveyed

Hypothesis
NUMBER OF INTERVALS SPANNED:
5, 6 or 7

2nd Pillar: Synthetic risk indicator

Hypothesis
NUMBER OF INTERVALS SPANNED:

5 risk classes:
- Low
- Medium-Low
- Medium
- Medium-High
- High

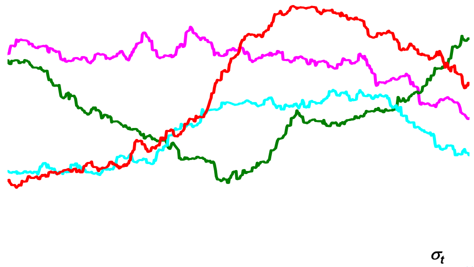
6 risk classes:
- Low
- Medium-Low
- Medium
- Medium-High
- High
- Very High

7 risk classes:
- Very Low
- Low
- Medium-Low
- Medium
- Medium-High
- High
- Very High

2nd Pillar: Synthetic risk indicator

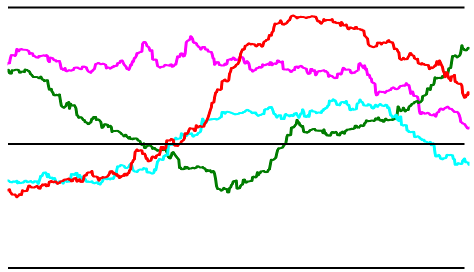
RISK TARGET PRODUCT | **BENCHMARK PRODUCT** | **RETURN TARGET PRODUCT**

Products with the same risk budget
must have the same degree of risk



2nd Pillar: Synthetic risk indicator

Volatility intervals have to be suitably calibrated
in order to avoid wrong risk representations



2nd Pillar: Synthetic risk indicator

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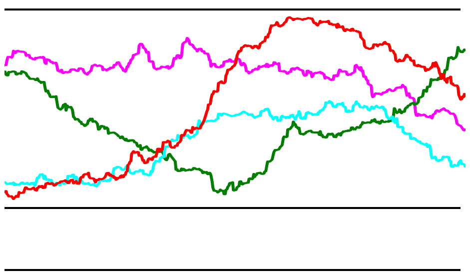


2nd Pillar: Synthetic risk indicator

Volatility intervals have to be suitably calibrated
in order to avoid wrong risk representations

Moreover, the optimal set of volatility intervals
has to be consistent with the principle:

+ RISK + LOSSES

**VOLATILITY INTERVALS MUST HAVE
AN INCREASING WIDTH**

2nd Pillar: Synthetic risk indicator

How to define a suitable volatility grid

Minimizing the chance for an asset manager
of overcoming not intentionally his risk budget,
i.e. the volatility interval
(so-called "*management failure*")

2nd Pillar: Synthetic risk indicator

How to define a suitable volatility grid

Minimizing the chance for an asset manager of overcoming not intentionally his risk budget, i.e. the volatility interval (so-called "*management failure*")

The optimal set of volatility intervals for a given *n*-tuple of risk classes requires to solve a stochastic NLP problem (i.e. minimize the chance of a "*management failure*")

2nd Pillar: Synthetic risk indicator

How to define a suitable volatility grid

In order to analyze the *management failures*, (i.e.: to specify and solve the SNLP problem)….

1st INTUITION

it has to be studied the behavior of an automatic asset manager that has a specific risk budget, identified by a given volatility interval

2nd Pillar: Synthetic risk indicator

How to define a suitable volatility grid

In order to analyze the *management failures*, (i.e.: to specify and solve the SNLP problem)….

2nd INTUITION

volatility prediction intervals have to be determined, in order to measure the ability of the automatic asset manager to remain within his risk budget

2nd Pillar: Synthetic risk indicator

How to define a suitable volatility grid

In order to analyze the *management failures*, (i.e.: to specify and solve the SNLP problem)….

3rd INTUITION

the optimal set of volatility intervals must allow a similar number of "*management failures*" to any automatic asset managers (despite his belonging to different risk classes)

NO INCENTIVE TO CHOOSE A SPECIFIC CLASS

2nd Pillar: Synthetic risk indicator

The stochastic non linear programming problem

optimal set of volatility intervals

Let $n \in \mathbb{N}$ be the number of volatility intervals (so-called "*n*-tuple of risk classes")

Then, the optimization problem is twofold:

1. find the optimal number of intervals: n^*
2. for $n=n^*$ minimize the management failures

as defined below:

$$\min_{\sigma_1<\sigma_2<\cdots<\sigma_n}\left(\max_{i=1,\ldots,n} mf_i\right)$$

$$s.t.\ mf_i \approx mf_{i-1}$$

2nd Pillar: Synthetic risk indicator

The stochastic non linear programming problem

optimal set of volatility intervals

1st INTUITION **2nd INTUITION** **3rd INTUITION**

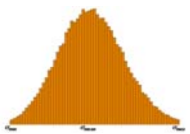
2nd Pillar: Synthetic risk indicator

1st INTUITION

Automatic Asset Manager

Hypothesis:

Stochastic volatility model where the automatic asset manager is "mean-reverting":



The automatic asset manager:

- has no systematic preference for upwards or downwards deviations from the mean ➔ symmetric distribution for the volatility
- in order to minimize the migration risk, keeps the product volatility far from the bounds of the interval ➔ probability decay over the tails

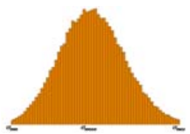
2nd Pillar: Synthetic risk indicator

1st INTUITION

Automatic Asset Manager

Hypothesis:

Stochastic volatility model where the automatic asset manager is "mean-reverting":



A proper definition of the parameters for the following pair of SDEs:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^{(1)}$$

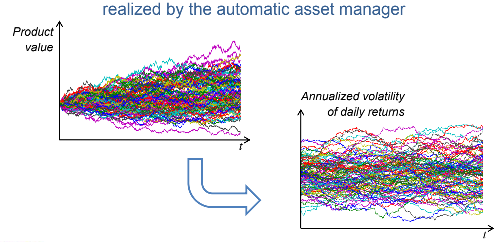
$$d\sigma_t^2 = \kappa(\vartheta - \sigma_t^2)dt + v_t dW_t^{(2)}$$

2nd Pillar: Synthetic risk indicator

1st INTUITION

Automatic Asset Manager

Simulation of the trajectories of the volatility realized by the automatic asset manager



2nd Pillar: Synthetic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals

Definition 1

An a-confident volatility prediction interval is defined by the pair $[\sigma_{min,t}, \sigma_{max,t}]$ s.t.:

$$\Pr(\sigma_{t,min} \le \sigma_t^{AAM} \le \sigma_{t,max}) = \alpha$$

*where σ_t^{AAM} is the annualized daily returns volatility realized by the automatic asset manager at day **t** based on the last 252 product's daily returns.*

Definition 2

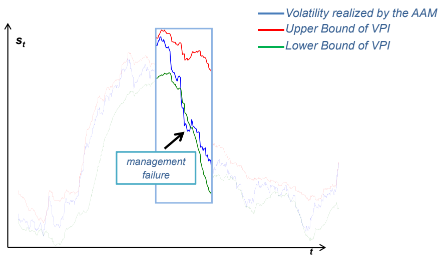
*A "management failure" is said to occur at day **t** if either $\sigma_t^{AAM} > \sigma_{t,max}$ or $\sigma_t^{AAM} < \sigma_{t,min}$*

2nd Pillar: Synthetic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals

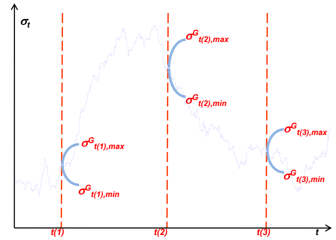


2nd Pillar: Synthetic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals



Hypothesis:

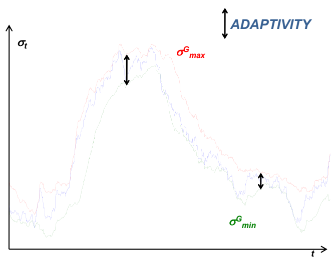
Adaptive GARCH diffusive models to measure the ability of the automatic asset manager to remain within his risk budget

2nd Pillar: Synthetic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals



Hypothesis:

Adaptive GARCH diffusive models to measure the ability of the automatic asset manager to remain within his risk budget

2nd Pillar: Synthetic risk indicator

Management Failures

Diffusive GARCH Implementation

from the M-GARCH(1,1)

$$\begin{cases} X_k - X_{k-1} = \gamma\cdot(\eta - X_{k-1}) + \sigma_k \tilde{Z}_k \\ \text{and} \\ \ln\sigma^2_{k+1} - \ln\sigma^2_k = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma^2_k + \beta_1^{(k)}\ln Z_k^2 \\ \text{or, equivalently} \\ \ln\sigma^2_{k+1} - \ln\sigma^2_k = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma^2_k + 2\beta_1^{(k)}\ln|Z_k| \end{cases}$$

$\tilde{Z}_k$ and Z_k are i.i.d. N(0,1)

Weak Convergence Theorem of Discrete Markov Chains to Diffusions

$$dX_t = \varphi(\mu - X_t)dt + \sigma_t dW_t$$

$$d\ln\sigma_t^2 = (\beta_0 + 2\beta_1 E(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2)dt + 2|\beta_1|\sqrt{Var(\ln|Z_t|)}dW_t^*$$

Z_t is N(0,1)

O.U. process

$$\ln\sigma_t^2 \sim N\left(\left(\ln\sigma_s^2 + \frac{\beta_0 + 2\beta_1 E(\ln|Z_t|)}{(\beta_1 - 1)}\right)e^{(\beta_1-1)(t-s)} - \frac{\beta_0 + 2\beta_1 E(\ln|Z_t|)}{(\beta_1 - 1)}; \sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2 \left(e^{2(\beta_1-1)(t-s)} - 1\right)}{2(\beta_1 - 1)}} \right)$$

2nd Pillar: Synthetic risk indicator

Management Failures

Diffusive GARCH Implementation

maximum likelihood estimation

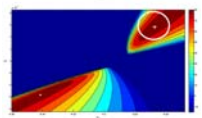
the likelihood function

$$L(w; \underline{\beta}) = \prod_{k=2}^{K} [\ldots]$$

where: $\underline{\beta} = (\beta_0, \beta_1)$

***K** = number of observations of annualized daily volatility used to estimate the parameters*

β_0 and β_1 estimates



REMARK

Adaptivity of Diffusive GARCH allows to work with poorer filtrations:

***K** reasonably small (around 60)*

2nd Pillar: Synthetic risk indicator

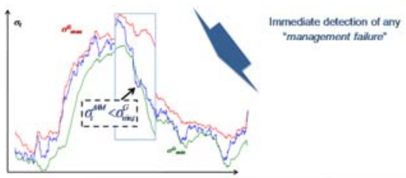
Management Failures

Diffusive GARCH Implementation

the estimated parameters enter in the bounds of the volatility prediction interval

$\sigma^G_{t,min} = \ldots$

$\sigma^G_{t,max} = \ldots$



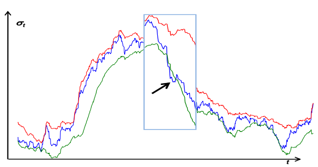
2nd Pillar: Synthetic risk indicator

3rd INTUITION

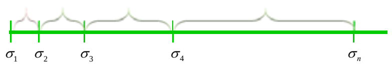
NO INCENTIVE TO CHOOSE A SPECIFIC CLASS

$$mf_1 \approx mf_2 \approx mf_3 \approx mf_4 \approx \ldots \approx mf_n$$

Management Failures



Volatility Prediction Intervals



Suitable volatility grid

2nd Pillar: Synthetic risk indicator

The stochastic non linear programming problem

Solution to step 1

The higher is *n* the smaller will be the average width of the volatility intervals and the lower is the average number of the management failures

***n*=7**

2^{nd} Pillar: Synthetic risk indicator

The stochastic non linear programming problem

Solution to step 2

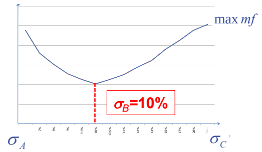
LEMMA (for two consecutive intervals)

Let σ_A and σ_C be two known volatilities with $\sigma_A<\sigma_C$. Then, the value of σ_B s.t.:

$$\min_{\sigma_B}\left(\max\left\{mf_{[\sigma_A,\sigma_B]}, mf_{[\sigma_B,\sigma_C]}\right\}\right)$$

is: $\sigma_B=\sqrt{\sigma_A\sigma_C}$ or, equivalently: $\frac{\sigma_B}{\sigma_A}=\frac{\sigma_C}{\sigma_B}=m$

where *m* is called "*multiplier*".



2^{nd} Pillar: Synthetic risk indicator

The stochastic non linear programming problem

COROLLARY

Let $[\sigma_1,\sigma_2]$ and $[\sigma_3,\sigma_4]$ be two volatility intervals having the same multiplier *m*, i.e.:

$$m=\frac{\sigma_2}{\sigma_1}=\frac{\sigma_4}{\sigma_3}$$

then, the two intervals have the same number of "*management failures*", i.e.:

$$mf_1=mf_2$$

where $mf_i,\ i=1,2$ is the total number of management failures occurred to the automatic asset manager of the i^{th} volatility interval.

2^{nd} Pillar: Synthetic risk indicator

The stochastic non linear programming problem

the *1st* and the *nth* interval cannot respect the multiplier

the *1st* and the *nth* interval must be chosen looking at exogenous information

2^{nd} Pillar: Synthetic risk indicator

The stochastic non linear programming problem

ASSUMPTIONS

25% AS THE LOWER BOUND OF THE LAST VOLATILITY INTERVAL

…corresponding to a percentage loss of about 50% of the invested capital over a 1-year time horizon

0.25% AS THE UPPER BOUND OF THE FIRST VOLATILITY INTERVAL

…corresponding to typical results of monetary markets instruments

2^{nd} Pillar: Synthetic risk indicator

The stochastic non linear programming problem

the optimization problem becomes:

given *n**=7:

$$\min_{\sigma_2<\sigma_3<\cdots<\sigma_7}\left(\max_{i=2,\ldots,6} mf_i\right)$$

$$s.t.\ mf_i \approx mf_{i-1}$$

with: σ_2=0.25% σ_7=25%

2^{nd} Pillar: Synthetic risk indicator

Suitable volatility grid

OUTPUT

$m^*=2.5$

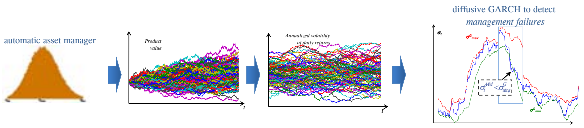
Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
Very Low	0.01%	0.24%
Low	0.25%	0.63%
Medium-Low	0.64%	1.59%
Medium	1.60%	3.99%
Medium-High	4.00%	9.99%
High	10.00%	24.99%
Very High	25.00%	>25.00%

The optimal set of volatility intervals is consistent with the principle: **+ RISK + LOSSES**

2^{nd} Pillar: Synthetic risk indicator

Definition of a suitable volatility grid

summarizing:



given $n^*=7$:

$$\min_{\sigma_2<\sigma_3<\cdots<\sigma_7}\left(\max_{i=2,\ldots,6} mf_i\right)$$

$$s.t.\ mf_i \approx mf_{i-1}$$

with: $\sigma_2=0.25\%$ $\sigma_7=25\%$

$$\frac{\sigma_2}{\sigma_1}=\frac{\sigma_3}{\sigma_2}=\frac{\sigma_4}{\sigma_3}=\frac{\sigma_5}{\sigma_4}=\cdots=\frac{\sigma_n}{\sigma_{n-1}}=m$$

$m^*=2.5$

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
Very Low	0.01%	0.24%
Low	0.25%	0.63%
Medium-Low	0.64%	1.59%
Medium	1.60%	3.99%
Medium-High	4.00%	9.99%
High	10.00%	24.99%
Very High	25.00%	>25.00%

Syllabus

Preliminaries

Three-pillars approach:

1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

3^{rd} Pillar: The recommended Investment horizon

RISK TARGET PRODUCT | **BENCHMARK PRODUCT** | **RETURN TARGET PRODUCT**

The recommended investment time horizon

for performance target products the recommended minimum investment horizon is inherent to their financial engineering, as the recommended investment horizon is equal to the period of validity (or the time to maturity) of their target

The payoff at maturity uniquely identifies the time when the potential returns are optimized

3rd Pillar : The recommended Investment horizon

RISK TARGET PRODUCT | BENCHMARK PRODUCT | **RETURN TARGET PRODUCT**

The recommended investment time horizon

The use of solutions aimed at ensuring the liquidity and/or marketability of a return target product changes its risk-return profile and its recommended investment time horizon

The event to study from a probabilistic point of view transforms into:

> The investment recovers the initial costs and off-sets the running costs at least once

that can be calculated through the concept of

First Passage Time

The "minimum" recommended investment time horizon

3rd Pillar : The recommended Investment horizon

RISK TARGET PRODUCT | BENCHMARK PRODUCT | RETURN TARGET PRODUCT

The "minimum" recommended investment time horizon

For risk target products, the natural way to define a cost recovery event is also:

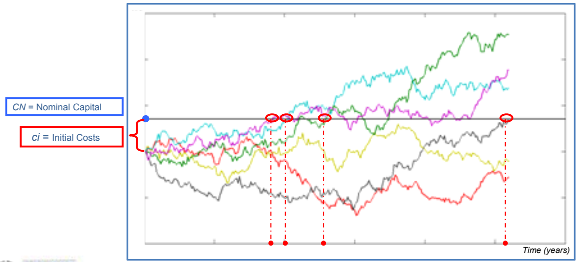
> The investment recovers the initial costs and off-sets the running costs at least once

that can be calculated through the concept of

First Passage Time

3rd Pillar : The recommended Investment horizon

First Passage Time:
First time (expressed in years) such that the value of the Invested Capital (CI) recovers the initial costs and off-sets the running costs.



3rd Pillar : The recommended Investment horizon

The probability of the event:

> The investment recovers the initial costs and off-sets the running costs at least once

given a confidence level α, uniquely identifies a time T^* on the cumulative distribution function of the first passage times, i.e.:

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \leq T\right] = \alpha \right\}$$

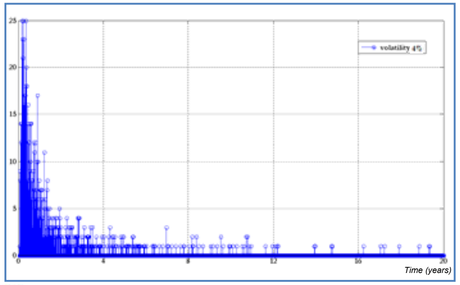
where

$$t^* = \inf\left[t \in \Re^+ : CI_t > CN\right]$$

is the first passage time

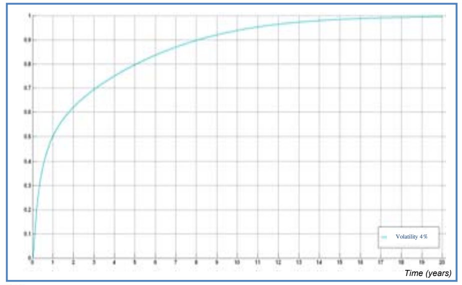
3rd Pillar : The recommended Investment horizon

1. Calculation of the probability distribution of the first passage times:



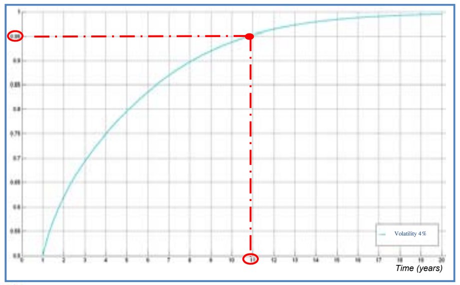
3rd Pillar : The recommended Investment horizon

2. Derivation of the cumulative distribution function of the first passage times:



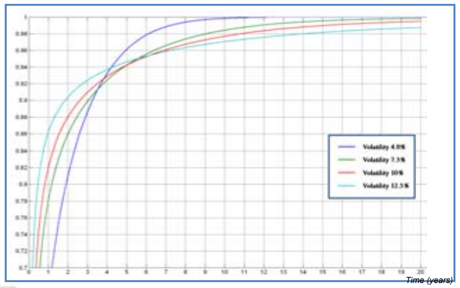
3rd Pillar : The recommended Investment horizon

3. The confidence level α uniquely identifies T^* on the cumulative distribution function of the first passage times:



3rd Pillar : The recommended Investment horizon

When many probability distribution functions are considered, letting varying volatilities and costs, the problem of correctly identifying a set of minimum thresholds arises:



3rd Pillar : The recommended Investment horizon

Anyway, the recommended minimum investment time horizon…

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \leq T\right] = \alpha \right\}$$

…. Must be coherent with the principle

+ VOLATILITY + TIME HORIZON

The correct way to solve the problem is to set up an operative procedure to select properly each threshold according to the above principle

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

First passage times for the break-even barrier are monitored at infinitesimal time intervals:

$dt \to 0$

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \le T\right] = \alpha \right\}$$

$$\mathrm{P}\left[t^* \le T\right] = N\left(d_2\left(\frac{CI_0}{CN}\right)\right) + \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \cdot N\left(-d_2\left(\frac{CN}{CI_0}\right)\right)$$

$$d_2(x) = \frac{\log x + \left(\bar{r} - cr - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

Asymptotic properties: $T \to \infty$

cr : recurrent costs as a fixed %

$dt \to 0$

$$\lim_{T\to\infty} \mathrm{P}\left[t^* \le T\right] = \begin{cases} 1 & if \quad (\bar{r}-cr) \ge \frac{1}{2}\sigma^2 \\ \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} & if \quad (\bar{r}-cr) < \frac{1}{2}\sigma^2 \end{cases}$$

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

Under our assumptions:

$dt \to 0$

$$\lim_{T\to\infty} \mathrm{P}\left[t^* \le T\right] = \begin{cases} 1 & if \quad (\bar{r}-cr) \ge \frac{1}{2}\sigma^2 \\ \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} & if \quad (\bar{r}-cr) < \frac{1}{2}\sigma^2 \end{cases}$$

For a given level of costs, it is possible to analytically derive the connection between volatility and time horizon

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

FIRST ORDER SENSITIVITY ANALYSIS

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1}\right)$$

FIRST ORDER ASYMPTOTIC CONDITION

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1}\right)$$

1. $(\bar{r}-cr) > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $(\bar{r}-cr) \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0$

The existence of two alternative states of nature requires to verify whether both of them make sense in financial terms under the risk-neutral measure.

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0$

Being running costs a specific feature of any financial product they would interfere with the task of identifying which of the two conditions has a sound financial meaning. Therefore, they will be temporarily neglected.

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0$

Since it is safe to assume a positive interest rate *r* in financial markets, only condition 1. correctly captures the connection between volatility and time horizon.

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0$

As $T \to \infty$ condition 1. implies that the cumulative distribution function P is a strictly decreasing function of the volatility, i.e.:

$$\forall \sigma_i, \sigma_j \in \Re^+, \sigma_j > \sigma_i \Rightarrow \mathrm{P}(\sigma_j) < \mathrm{P}(\sigma_i)$$

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$cr = 0$

1. $\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0$

In other words, for a given confidence level, as the volatility grows, the recommended investment time horizon increases as well:

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d^2P}{d\sigma^2} = \frac{4}{\sigma^4}(\bar{r}-cr)\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1}\cdot\left[1+\frac{4(\bar{r}-cr)}{\sigma^2}\ln\left(\frac{CN}{CI_0}\right)\right]$$

$(\bar{r}-cr) > 0 \Rightarrow \frac{d^2P}{d\sigma^2} > 0$ ⇒ SECOND ORDER ASYMPTOTIC CONDITION

Second Order Sensitivity Analysis

3rd Pillar : The recommended Investment horizon

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

1. $\begin{cases}(\bar{r}-cr) > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0 \\ (\bar{r}-cr) > 0 \Rightarrow \frac{d^2P}{d\sigma^2} > 0\end{cases}$

2. ~~$(\bar{r}-cr) \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0$~~

$\exists T^* \in [0,\infty[: \frac{dP}{d\sigma} = 0$

Summarizing the results of the asymptotic analysis in continuous time:
- As T →∞, for given a confidence level, more volatility implies a larger recommended investment time horizon
- It is always possible to find a minimum and finite time T^*, beyond which the strong condition

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

holds

3rd Pillar : The recommended Investment horizon

DETERMINATION OF THE INVESTMENT TIME HORIZON

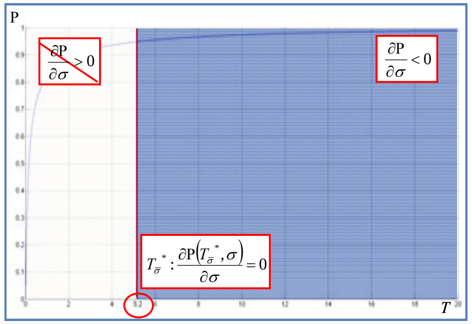
General Framework:

$\begin{cases} T \to \infty \\ dt \to 0 \\ P(\infty,\sigma) \\ 1. \begin{cases}(\bar{r}-cr) > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0 \\ (\bar{r}-cr) > 0 \Rightarrow \frac{d^2P}{d\sigma^2} > 0\end{cases}\end{cases} \Rightarrow \begin{cases} T \text{ finite} \\ dt \to 0 \\ P(T,\sigma) \\ (\bar{r}-cr) > 0 \Leftrightarrow \lim_{T\to\infty}\frac{\partial P(T,\sigma)}{\partial\sigma} < 0 \\ (\bar{r}-cr) > 0 \Rightarrow \lim_{T\to\infty}\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2} > 0\end{cases}$

Everything shown above also holds with *T* finite!

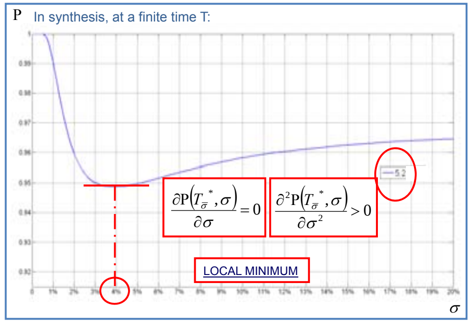
3rd Pillar : The recommended Investment horizon

DETERMINATION OF THE INVESTMENT TIME HORIZON



3rd Pillar : The recommended Investment horizon

DETERMINATION OF THE INVESTMENT TIME HORIZON

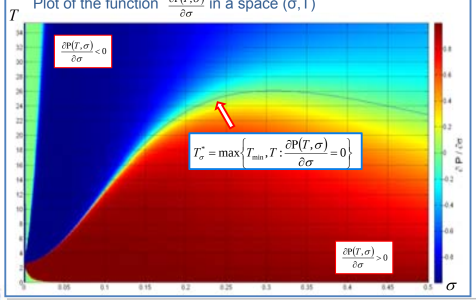


3rd Pillar : The recommended Investment horizon

DETERMINATION OF THE INVESTMENT TIME HORIZON

FIRST ORDER SENSITIVITY ANALYSIS

Plot of the function $\frac{\partial P(T,\sigma)}{\partial\sigma}$ in a space (σ,T)



3rd Pillar : The recommended Investment horizon

DETERMINATION OF THE INVESTMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS $\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$

Given the monotonicity condition of the probability distribution with respect to volatility, i.e.:

$$\forall \sigma_i, \sigma_j \in \Re^+, \sigma_j > \sigma_i \Rightarrow P(\infty,\sigma_j) < P(\infty,\sigma_i)$$

In order to fulfill this condition, it's necessary to restrict the analysis in the region where the probability function is strictly increasing, i.e.:

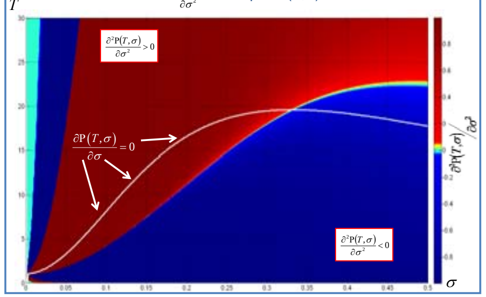
$$\left.\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}\right|_{T=T^*_\sigma} > 0 \Rightarrow T^*_\sigma \text{ increasing}$$

~~$\left.\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}\right|_{T=T^*_\sigma} < 0 \Rightarrow T^*_\sigma \text{ decreasing}$~~

3rd Pillar : The recommended Investment horizon

DETERMINATION OF THE INVESTMENT TIME HORIZON

Plot of the function $\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$ in a space (σ,T)



3rd Pillar : The recommended Investment horizon

DETERMINATION OF THE INVESTMENT TIME HORIZON

Plot of the function $\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$ in a space (σ,T)