CONSOB
COMMISSIONE NAZIONALE PER LE SOCIETÀ E LA BORSA

Integrated risk measurement and representation for non-equity products: how to frame material risks over the time horizon of the investment

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Risk magazine presents
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Syllabus

Preliminaries

Three-pillars approach:

1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

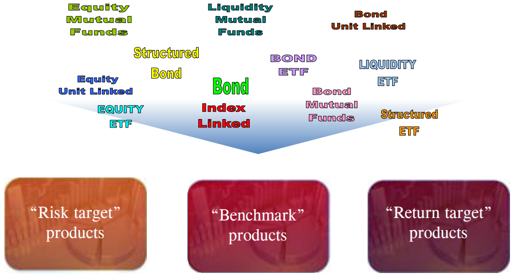
Preliminaries

Non-equity Investment products: definition

Equity Mutual Funds
Liquidity Mutual Funds
Certificates
Bond Unit Linked
Covered Warrants
Structured Bond
Equity Unit Linked
BOND ETF
LIQUIDITY ETF
Bond
Bond Mutual Funds
EQUITY ETF
Index Linked
Structured ETF

Preliminaries

Non-equity Investment products should be classified according to their financial characteristics and not by **"labels"** that are assigned by the issuer and/or by the European regulatory framework.



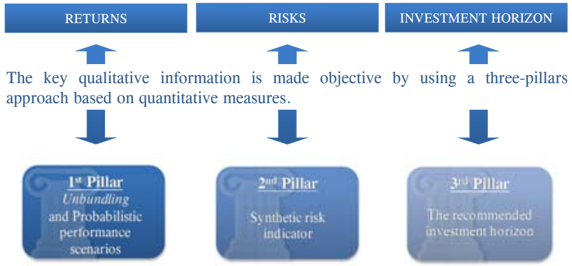
Preliminaries

The transparency on the risk profile of non-equity investment products is based on three synthetic indicators (three pillars) – defined through the development of specific quantitative methods – in order to allow investors to take informed investment decisions.

Traditional narrative description of all possible risks associated with a predefined **"label"**

Synthetic indicators robust, objective and backward verifiable

Preliminaries



The key qualitative information is made objective by using a three-pillars approach based on quantitative measures.

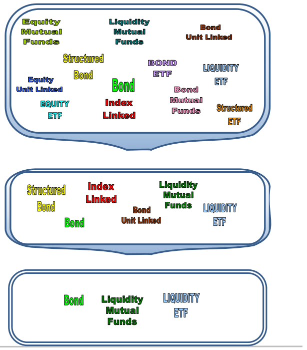
Preliminaries

Investor decisions as a sequential filtering problem:

Time goal:
liquidity/investment horizon
INVESTMENT HORIZON
(less than 3 years)

Risk profile:
risk limit in terms of downside
RISKS
(medium-low)

Return goal:
target returns
RETURNS
(maximum return)



Syllabus

Preliminaries

Three-pillars approach:

1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

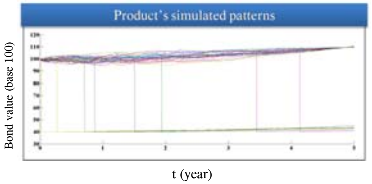
1st Pillar

In "return target" products (e.g. corporate bonds) the connection between the pricing at time zero and the pricing at maturity is evident, as the probability table is a necessary step to obtain the unbundling of the price of the product at time 0.

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

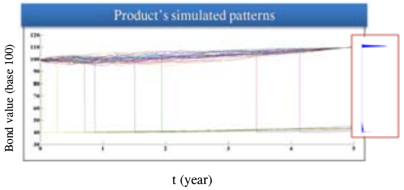
5 year fixed-rate bond


Euribor's simulated patterns


Product's simulated patterns

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

The final values of the bond at the end of the 5th year provide the probability distribution of potential returns (so-called *pricing* at maturity).


Product's simulated patterns

Possible outcomes
Pricing at maturity

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

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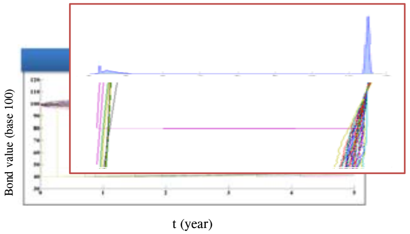
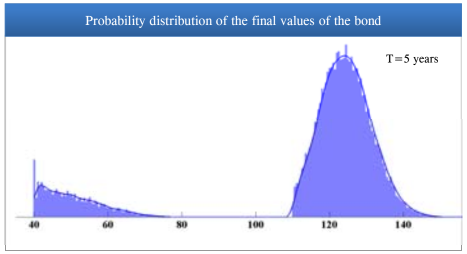


Possible outcomes
Pricing at maturity

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

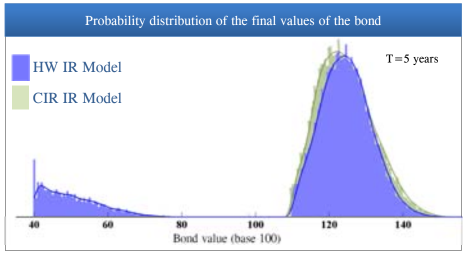
The informative content of the entire probability distribution is very complex to handle for the average retail investor.


Probability distribution of the final values of the bond

Possible outcomes
Pricing at maturity

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

The shape of the probability distribution of potential returns is obviously dependent from the model's assumption.


Probability distribution of the final values of the bond

1st Pillar
Unbundling
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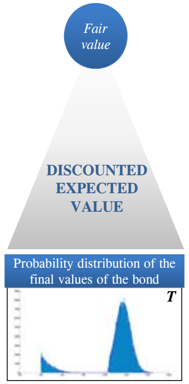
performance
scenarios

STANDARD SOLUTION

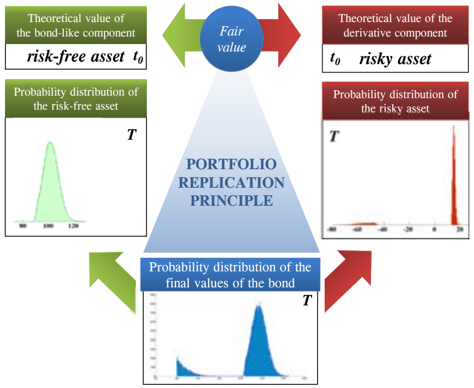

Fair value

DISCOUNTED EXPECTED VALUE

Probability distribution of the final values of the bond

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

CONSOB REGULATION


Theoretical value of the bond-like component

risk-free asset t_0

Fair value

Theoretical value of the derivative component

t_0 ***risky asset***

Probability distribution of the risk-free asset

Probability distribution of the risky asset

PORTFOLIO REPLICATION PRINCIPLE

Probability distribution of the final values of the bond

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

CONSOB REGULATION

Theoretical value of the bond-like component
risk-free asset t_0

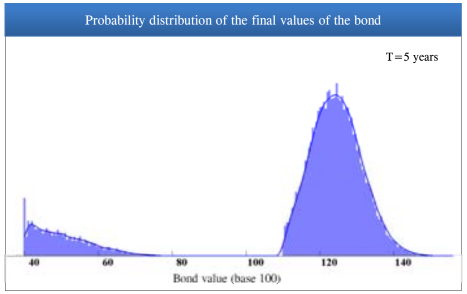
Fair value

Theoretical value of the derivative component
t_0 ***risky asset***

A	Theoretical value of the bond-like component	…
B	Theoretical value of the derivative component	…
C = A + B	*Fair value*	…
D	Explicit costs	…
E	Implicit costs	…
F = C + D + E	Issue price	100

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

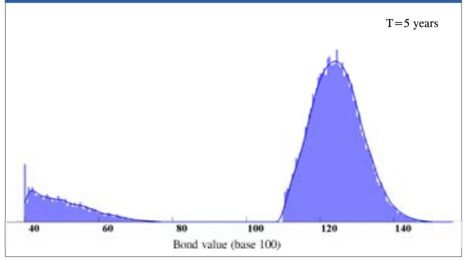
CONSOB REGULATION


Probability distribution of the final values of the bond

It's interesting to explore a different representation of the information contained in the probability distribution which could be useful for the average investor

CONSOB REGULATION

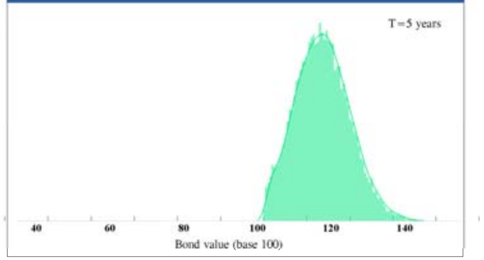
Probability distribution of the final values of the bond



In order to provide the investor with a representation fair, easy to understand and resilient to the model's risk, a simple rescaling with respect to the risk-neutral measure numeraire is presented

CONSOB REGULATION

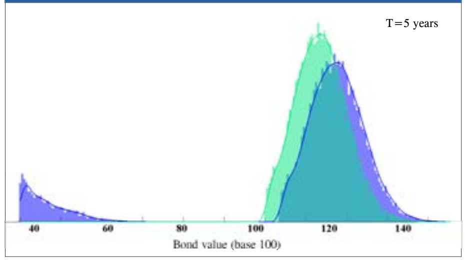
Probability distribution of the cash account (risk neutral numeraire)



In order to provide the investor with a representation fair, easy to understand and resilient to the model's risk, a simple rescaling with respect to the risk-neutral measure numeraire is presented

CONSOB REGULATION

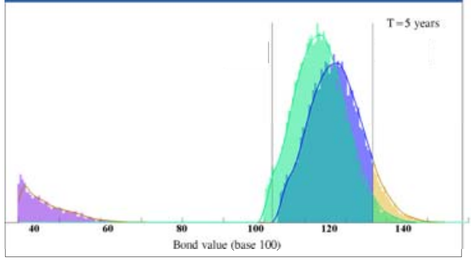
Probability distribution of the final values of the bond and the cash account



The superimposition of the product's probability distribution with the cash account naturally defines three different events which are effectively meaningful for the investor.

CONSOB REGULATION

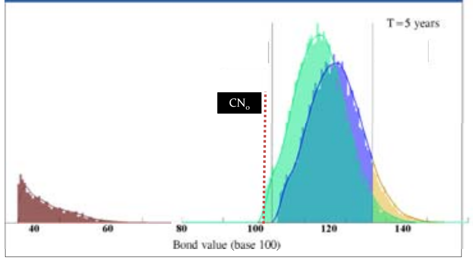
Probability distribution of the final values of the bond and the cash account



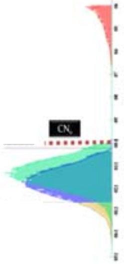
The performance is lower than the risk-free asset	The performance is positive and in line with the risk-free asset	The performance is positive and higher than the risk-free asset

CONSOB REGULATION

Probability distribution of the final values of the bond and the cash account



The performance is negative	The performance is positive and lower than the risk-free asset	The performance is positive and in line with the risk-free asset	The performance is positive and higher than the risk-free asset



SCENARIOS	PROBABILITY	MEDIAN VALUES
The performance is negative	%	€
The performance is positive but lower than the risk-free asset	%	€
The performance is positive and in line with the risk-free asset	%	€
The performance is positive and higher than the risk-free asset	%	€

Connection between the pricing at time zero and the pricing at the end of recommended investment horizon

Time Zero

Financial investment table

A	Theoretical value of the bond-like component	…
B	Theoretical value of the derivative component	…
C = A + B	*Fair value*	…
D	Explicit costs	…
E	Implicit costs	…
F = C + D + E	Issue price	100

End of the recommended investment horizon

Table of probabilistic performance scenarios

SCENARIOS	PROBABILITY	MEDIAN VALUES
The performance is negative	%	€
The performance is positive but lower than the risk-free asset	%	€
The performance is positive and in line with the risk-free asset	%	€
The performance is positive and higher than the risk-free asset	%	€

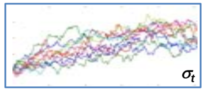
1:1 Relationship

Model Risk Assessment

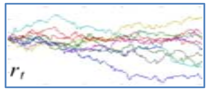
The recommended time horizon has a significant influence on the choice of the model

For Time Horizons greater than 1 year…..

I



II

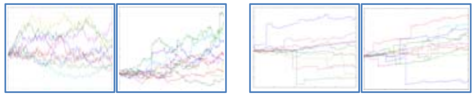


Model Risk Assessment

The recommended time horizon has a significant influence on the choice of the model

Many possible choices…

I	II	III	IV
HESTON Stochastic Volatility Model	MERTON Jump Diffusion Model	CARR MADAN CHANG Variance Gamma Model	BARNDORFF NIELSEN NIG Model

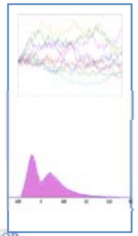


1st Pillar

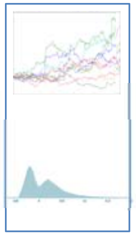
1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Step 2: Calculation of the Probability Distribution of the Invested Capital at the end of recommended time horizon

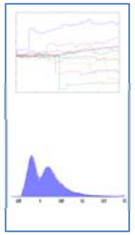
Heston



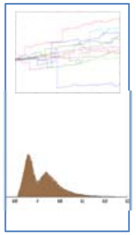
Merton



V G



NIG



1st Pillar

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Unbundling
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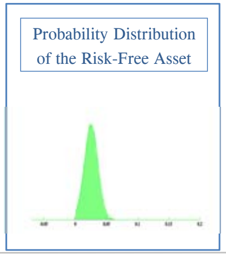
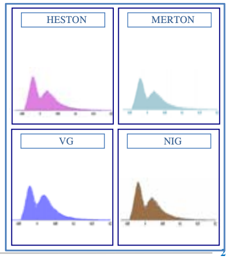


Probability Distribution of the Risk-Free Asset



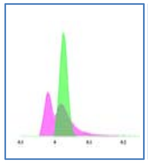
1st Pillar

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

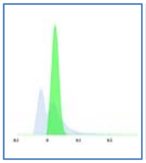
Step 3: Probabilistic comparison with the Risk-Free Asset

Analysing the probability distributions…

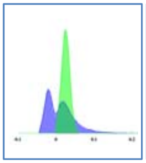
Heston



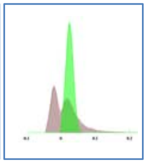
Merton



V G



NIG



1st Pillar

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

… the following output is obtained:

Heston

Scenarios	Proba bility	Median Values
The performance is negative	46.61 %	€ 90.50
The performance is positive but lower than the risk-free asset	3.39%	€ 101.26
The performance is positive and in line with the risk-free asset	33.28 %	€ 112.19
The performance is positive and higher than the risk-free asset	16.72 %	€ 139.93

Merton

Scenarios	Proba bility	Median Values
The performance is negative	42.695 %	€ 89.26
The performance is positive but lower than the risk-free asset	4.74%	€ 102.54
The performance is positive and in line with the risk-free asset	35.7%	€ 110.09
The performance is positive and higher than the risk-free asset	16.86 %	€ 142.65

V G

Scenarios	Proba bility	Median Values
The performance is negative	43.91 %	€ 91.25
The performance is positive but lower than the risk-free asset	5.23%	€ 102.1
The performance is positive and in line with the risk-free asset	36.8%	€ 109.24
The performance is positive and higher than the risk-free asset	14.06 %	€ 141.77

NIG

Scenarios	Proba bility	Median Values
The performance is negative	48.1%	€ 93.40
The performance is positive but lower than the risk-free asset	2.6%	€ 101.91
The performance is positive and in line with the risk-free asset	34.28 %	€ 114.23
The performance is positive and higher than the risk-free asset	15.02 %	€ 142.13

1st Pillar

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: $|\Delta| < 4.7\%$

Heston

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V G

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1st Pillar

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: $|\Delta| < 2.7\%$

Heston

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1st Pillar

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: $|\Delta| < 3.7\%$

Heston

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The performance is negative	46.61 %	€ 90.50
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1st Pillar

1st Pillar
Unbundling
and Probabilistic
performance
scenarios

Step 3: Probabilistic comparison with the Risk-Free Asset

Assessing the model risk: $|\Delta| < 1.2\%$

Heston

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Syllabus

Preliminaries

Three-pillars approach:

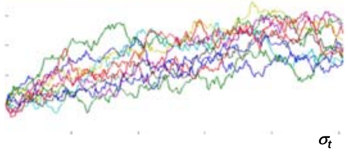
1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

2nd Pillar
Synthetic risk indicator

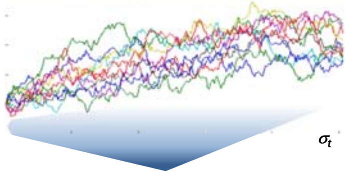
Volatility of the product's simulated returns



Volatility is the most immediate risk measure and
it has a one-to-one relationship with whatever loss measure
(VaR, ES, etc.)

2nd Pillar
Synthetic risk indicator

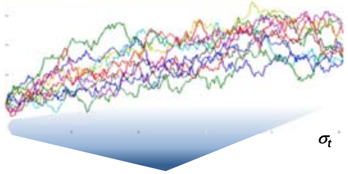
Volatility of the product's simulated returns



DEGREE OF RISK

2nd Pillar
Synthetic risk indicator

Volatility of the product's simulated returns



DEGREE OF RISK

MEASUREMENT:
product's positioning inside
a grid of *n* volatility intervals

REPRESENTATION:
mapping of any volatility interval into a
corresponding qualitative risk class

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
low	$\sigma_{1,min}$	$\sigma_{1,max}$
medium-low	$\sigma_{2,min}$	$\sigma_{2,max}$
medium	$\sigma_{3,min}$	$\sigma_{3,max}$
medium -high	$\sigma_{4,min}$	$\sigma_{4,max}$
high	$\sigma_{5,min}$	$\sigma_{5,max}$
very high	$\sigma_{6,min}$	$\sigma_{6,max}$

2nd Pillar
Synthetic risk indicator

MEASUREMENT:
product's positioning inside
a grid of *n* volatility intervals

REPRESENTATION:
mapping of any volatility interval into a
corresponding qualitative risk class

Looking for the number of intervals
(so-called "*n*-tuple of risk classes")
allowing the best compromise between
investors' comprehension and
detail of the information conveyed

Hypothesis
NUMBER OF INTERVALS SPANNED:
5, 6 or 7

2nd Pillar
Synthetic risk indicator

Hypothesis
NUMBER OF INTERVALS SPANNED:

5 risk classes
- Low
- Medium-Low
- Medium
- Medium-High
- High

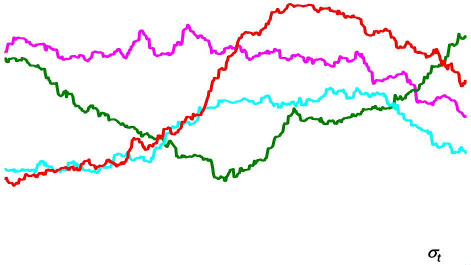
6 risk classes
- Low
- Medium-Low
- Medium
- Medium-High
- High
- Very High

7 risk classes
- Very Low
- Low
- Medium-Low
- Medium
- Medium-High
- High
- Very High

2nd Pillar
Synthetic risk indicator

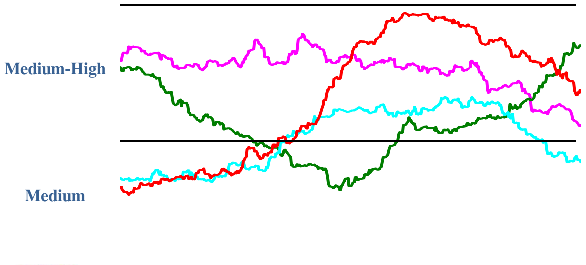
"Risk target" products
"Benchmark" products
"Return target" products

Products with the same risk budget
must have the same degree of risk



2nd Pillar
Synthetic risk indicator

Volatility intervals have to be suitably calibrated
in order to avoid wrong risk representations



2nd Pillar
Synthetic risk indicator

Volatility intervals have to be suitably calibrated
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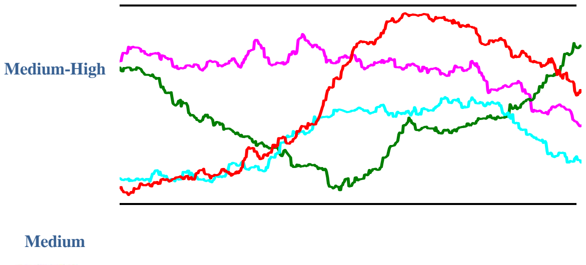


2nd Pillar
Synthetic risk indicator

Volatility intervals have to be suitably calibrated
in order to avoid wrong risk representations

Moreover, the optimal set of volatility intervals
has to be consistent with the principle:

+ RISK + LOSSES

**VOLATILITY INTERVALS MUST HAVE
AN INCREASING WIDTH**

How to define a suitable volatility grid

Minimizing the chance for an asset manager
of overcoming not intentionally his risk budget,
i.e. the volatility interval
(so-called "*management failure*")

How to define a suitable volatility grid

Minimizing the chance for an asset manager
of overcoming not intentionally his risk budget,
i.e. the volatility interval
(so-called "*management failure*")

The optimal set of volatility intervals for a given *n*-tuple
of risk classes requires to solve a stochastic NLP problem
(i.e. minimize the chance of a "*management failure*")

How to define a suitable volatility grid

In order to analyze the *management failures,* (i.e.: to specify and solve the SNLP problem)….

1st INTUITION

it has to be studied the behavior
of an automatic asset manager
that has a specific risk budget,
identified by a given volatility interval

How to define a suitable volatility grid

In order to analyze the *management failures,* (i.e.: to specify and solve the SNLP problem)….

2nd INTUITION

volatility prediction intervals have to be determined,
in order to measure the ability of the automatic asset
manager to remain within his risk budget

How to define a suitable volatility grid

In order to analyze the *management failures,* (i.e.: to specify and solve the SNLP problem)….

3rd INTUITION

the optimal set of volatility intervals must allow
a similar number of "*management failures*"
to any automatic asset managers
(despite his belonging to different risk classes)

NO INCENTIVE TO CHOOSE A SPECIFIC CLASS

The stochastic non linear programming problem

optimal set of volatility intervals

Let $n \in \mathbb{N}$ be the number of volatility intervals
(so-called "*n*-tuple of risk classes")

Then, the optimization problem is twofold:

1. find the optimal number of intervals: n^*
2. for $n=n^*$ minimize the management failures as defined below:

$$\min_{\sigma_1<\sigma_2<\cdots<\sigma_n}\left(\max_{i=1,\ldots,n} mf_i\right)$$

$$s.t.\ mf_i - mf_{i-1} \cong 0$$

The stochastic non linear programming problem

optimal set of volatility intervals

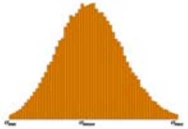
1st INTUITION **2nd INTUITION** **3rd INTUITION**

1st INTUITION

Automatic Asset Manager

Hypothesis:

Stochastic volatility model where the automatic
asset manager is "mean-reverting":



The automatic asset manager:

- has no systematic preference for upwards or downwards deviations from the mean ➔ symmetric distribution for the volatility
- in order to minimize the migration risk, keeps the product volatility far from the bounds of the interval ➔ probability decay over the tails

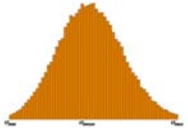
---

1st INTUITION

Automatic Asset Manager

Hypothesis:

Stochastic volatility model where the automatic
asset manager is "mean-reverting":



A proper definition of the parameters for the following pair of SDEs:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^{(1)}$$

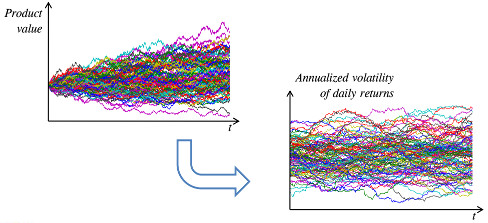
$$d\sigma_t^2 = \kappa(\vartheta - \sigma_t^2)dt + v_t dW_t^{(2)}$$

2nd Pillar
Systemic risk indicator

1st INTUITION

Automatic Asset Manager

Simulation of the trajectories of the volatility realized by the automatic asset manager



2nd Pillar
Systemic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals

Definition 1

An α-confident volatility prediction interval is defined by the pair $[\sigma_{\min,t},\sigma_{\max,t}]$ *s.t.:*

$$\Pr(\sigma_{t,\min} \le \sigma_t^{AAM} \le \sigma_{t,\max}) = \alpha$$

where σ_t^{AAM} *is the annualized daily returns volatility realized by the automatic asset manager at day* ***t*** *based on the last 252 product's daily returns.*

Definition 2

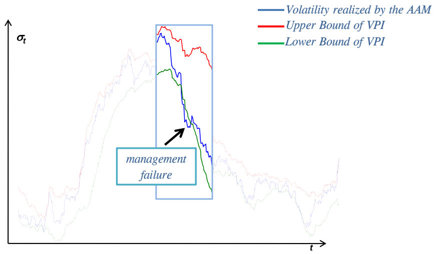
A "management failure" is said to occur at day ***t*** *if either*
$\sigma_t^{AAM} > \sigma_{t,\max}$ *or* $\sigma_t^{AAM} < \sigma_{t,\min}$

2nd Pillar
Systemic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals

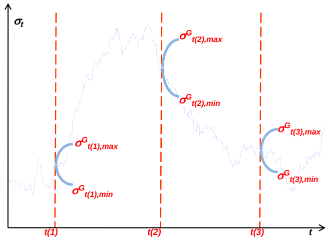


2nd Pillar
Systemic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals



Hypothesis:

Adaptive GARCH diffusive models to measure the ability of the automatic asset manager to remain within his risk budget

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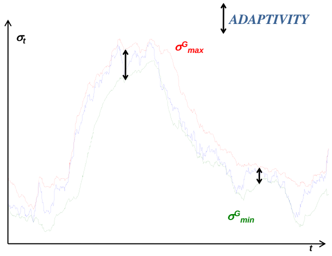
2nd Pillar
Systemic risk indicator

2nd INTUITION

Management Failures

Volatility Prediction Intervals



Hypothesis:

Adaptive GARCH diffusive models to measure the ability of the automatic asset manager to remain within his risk budget

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2nd Pillar
Systemic risk indicator

Management Failures

Diffusive GARCH Implementation

from the M-GARCH(1,1)

$$\begin{cases} X_k - X_{k-1} = \gamma\cdot(\eta - X_{k-1}) + \sigma_k \tilde{Z}_k \\ \text{and} \\ \ln\sigma^2_{k+1} - \ln\sigma^2_k = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma^2_k + \beta_1^{(k)}\ln Z^2_k \\ \text{or, equivalently} \\ \ln\sigma^2_{k+1} - \ln\sigma^2_k = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma^2_k + 2\beta_1^{(k)}\ln|Z_k| \end{cases}$$

$\tilde{Z}_k$ and Z_k are i.i.d. $N(0,1)$

Weak Convergence Theorem of Discrete Markov Chains to Diffusions

$$dX_t = q(\mu - X_t)dt + \sigma_t dW_t$$
$$d\ln\sigma^2_t = (\beta_0 + 2\beta_1 E(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma^2_t)\,dt + 2|\beta_1|\sqrt{Var(\ln|Z_t|)}dW^*_t$$

$Z_t \approx N(0,1)$

2nd Pillar
Systemic risk indicator

Management Failures

Diffusive GARCH Implementation

distributional properties of the S.D.E. of the M-GARCH(1,1)

$$d\ln\sigma^2_t = \left[\beta_0 + 2\beta_1\mathbf{E}(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma^2_t\right]dt + 2|\beta_1|\sqrt{Var(\ln|Z_t|)}dW^*_t$$

O.U. process

$$\ln\sigma^2_t \sim N\left(\frac{\left(\ln\sigma^2_s + \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)(t-s)} - \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)};}{\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2\left(e^{2(\beta_1-1)(t-s)}-1\right)}{2(\beta_1-1)}}}\right)$$

2nd Pillar
Systemic risk indicator

Management Failures

Diffusive GARCH Implementation

matching of the first two conditional moments

the discrete process becomes:

$$\begin{aligned}\ln\sigma^2_k - \ln\sigma^2_{k-1} = {} & \frac{[\beta_0+2\beta_1 E(\ln|Z_{k-1}|)]\left(e^{(\beta_1-1)}-1\right)}{\beta_1-1} - \\ & -2|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}}E\left(\ln|Z_{k-1}|\right) + \\ & +\left(e^{(\beta_1-1)}-1\right)\ln\sigma^2_{k-1} + \\ & +2|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}}\ln|Z_{k-1}|\end{aligned}$$

2nd Pillar
Systemic risk indicator

Management Failures

Diffusive GARCH Implementation

maximum likelihood estimation

the likelihood function

$$L\left(w;\underline{\beta}\right) = \prod_{k=2}^{K}\left[\frac{1}{|\beta_1|\sqrt{2\pi}}\sqrt{\frac{2(\beta_1-1)}{e^{2(\beta_1-1)}-1}}\cdot e^{\left(\frac{1}{2|\beta_1|}\sqrt{\frac{2(\beta_1-1)}{e^{2(\beta_1-1)}-1}}\cdot w_k\right)} \cdot e^{\left(-\frac{1}{2}\exp\left(\frac{1}{|\beta_1|}\sqrt{\frac{2(\beta_1-1)}{e^{2(\beta_1-1)}-1}}\cdot w_k\right)\right)}\right]$$

where: $\underline{\beta} := (\beta_0, \beta_1)$

$$w_k := y_k - \frac{(\beta_0 - 1.27\beta_1)(e^{(\beta_1-1)}-1)}{\beta_1-1} - 1.27|\beta_1|\sqrt{\frac{e^{2(\beta_1-1)}-1}{2(\beta_1-1)}} - (e^{(\beta_1-1)}-1)\ln\sigma^2_{k-1}$$

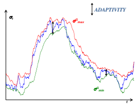
K *:= number of observations of annualized daily volatility used to estimate the parameters*

Management Failures

Diffusive GARCH Implementation

maximum likelihood estimation

REMARK



Adaptivity of Diffusive GARCH
allows to work with poorer filtrations:

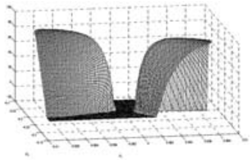
K reasonably small (around 60)

Management Failures

Diffusive GARCH Implementation

maximum likelihood estimation

shape of the log-likelihood function

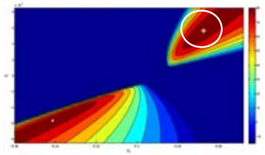


Management Failures

Diffusive GARCH Implementation

maximum likelihood estimation

β_o and β_1 estimates



Management Failures

Diffusive GARCH Implementation

the estimated parameters enter in the bounds
of the volatility prediction interval

$$\sigma^G_{t,\min} = e^{\frac{-z_{\frac{1+\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2\left(e^{2(\beta_1-1)}-1\right)}{2(\beta_1-1)}}+\left(\ln\sigma^2_{t-1}+\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)}}{2}}$$

$$\sigma^G_{t,\max} = e^{\frac{z_{\frac{1+\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2\left(e^{2(\beta_1-1)}-1\right)}{2(\beta_1-1)}}+\left(\ln\sigma^2_{t-1}+\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)}}{2}}$$

Management Failures

Diffusive GARCH Implementation

the α-confident volatility prediction interval becomes:

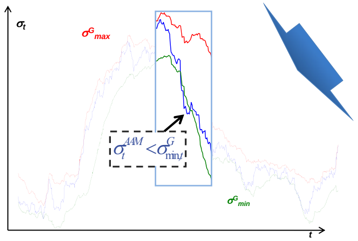
$$[\sigma^G_{t,\min},\sigma^G_{t,\max}]$$

i.e.:

$$\Pr(\sigma^G_{t,\min} \le \sigma^{AAM}_t \le \sigma^G_{t,\max}) = \alpha$$

Management Failures

Diffusive GARCH Implementation

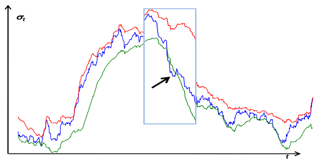


Immediate detection of any
"*management failure*"

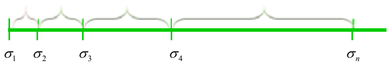
NO INCENTIVE TO CHOOSE A SPECIFIC CLASS

3rd INTUITION $mf_1 \simeq mf_2 \simeq mf_3 \simeq mf_4 \simeq \ldots \simeq mf_n$

Management Failures



Volatility Prediction Intervals



suitable volatility grid

The stochastic non linear programming problem

Solution to step 1

The higher is *n* the smaller will be the
average width of the volatility intervals
and the lower is the average number of
the management failures

$$n^* = 7$$

The stochastic non linear programming problem

Solution to step 2

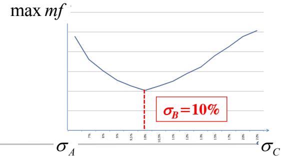
LEMMA (for two consecutive intervals)

Let σ_A and σ_C be two known volatilities with $\sigma_A < \sigma_C$. Then, the value of σ_B s.t.:

$$\min_{\sigma_B}\left(\max\left\{mf_{[\sigma_A,\sigma_B]}, mf_{[\sigma_B,\sigma_C]}\right\}\right)$$

is: $\sigma_B = \sqrt{\sigma_A\sigma_C}$ or, equivalently: $\frac{\sigma_B}{\sigma_A} = \frac{\sigma_C}{\sigma_B} = m$

where *m* is called "*multiplier*".



The stochastic non linear programming problem

COROLLARY (for two consecutive intervals)

Let $[\sigma_{1,}\sigma_2]$ and $[\sigma_{3,}\sigma_4]$ be two volatility intervals having the same multiplier m, i.e.:

$$m = \frac{\sigma_2}{\sigma_1} = \frac{\sigma_4}{\sigma_3}$$

then, the two intervals have the same number of "management failures", i.e.:

$$mf_1 = mf_2$$

where $mf_i,\ i = 1,2$ is the total number of management failures occurred to the automatic asset manager of the i-th volatility interval, that is the realized number of breaches of the diffusive GARCH-based volatility prediction intervals

---

The stochastic non linear programming problem

Generalizing for *n* consecutive intervals:

The lemma and the corollary can be applied iteratively leading to:

$$\frac{\sigma_2}{\sigma_1} = \frac{\sigma_3}{\sigma_2} \quad \frac{\sigma_3}{\sigma_2} = \frac{\sigma_4}{\sigma_3} \quad \frac{\sigma_4}{\sigma_3} = \frac{\sigma_5}{\sigma_4} \quad \cdots \quad \frac{\sigma_{n-1}}{\sigma_{n-2}} = \frac{\sigma_n}{\sigma_{n-1}}$$

$$\frac{\sigma_2}{\sigma_1} = \frac{\sigma_3}{\sigma_2} = \frac{\sigma_4}{\sigma_3} = \frac{\sigma_5}{\sigma_4} = \cdots = \frac{\sigma_n}{\sigma_{n-1}} = m$$

The stochastic non linear programming problem

the *1*st and the *n*th interval
cannot respect the multiplier

the *1*st and the *n*th interval must be chosen
looking at exogenous information

The stochastic non linear programming problem

ASSUMPTION

25% AS THE LOWER BOUND
OF THE LAST VOLATILITY INTERVAL

…corresponding to a percentage loss of about 50% of the invested capital over a 1-year time horizon

---

The stochastic non linear programming problem

ASSUMPTION

0.25% AS THE UPPER BOUND
OF THE FIRST VOLATILITY INTERVAL

…corresponding to typical results of monetary markets instruments

---

The stochastic non linear programming problem

the optimization problem becomes:

given $n^* = 7$:

$$\min_{\sigma_2<\sigma_3<\cdots<\sigma_7}\left(\max_{i=2,\ldots,6} mf_i\right)$$

$$s.t.\ mf_{i+1} - mf_i \simeq 0$$

$$with:\ \sigma_2 = 0.25\% \quad \sigma_7 = 25\%$$

Suitable volatility grid

OUTPUT

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
Very Low	0.01%	0.24%
Low	0.25%	0.59%
Medium-Low	0.6%	1.59%
Medium	1.60%	3.99%
Medium-High	4.00%	9.99%
High	10.00%	24.99%
Very High	25.00%	>25.00%

$m^* = 2.5$

Suitable volatility grid

REMARK

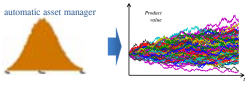
The optimal set of volatility intervals
is consistent with the principle:

+ RISK + LOSSES

**VOLATILITY INTERVALS HAVE
AN INCREASING WIDTH**

Definition of a suitable volatility grid

summarizing:



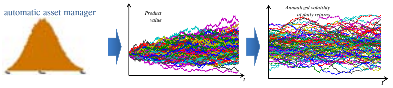
2nd Pillar

Synthetic risk indicator

Definition of a suitable volatility grid

summarizing:

automatic asset manager



2nd Pillar

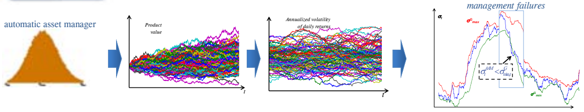
Synthetic risk indicator

Definition of a suitable volatility grid

summarizing:

automatic asset manager

diffusive GARCH to detect *management failures*



2nd Pillar

Synthetic risk indicator

Definition of a suitable volatility grid

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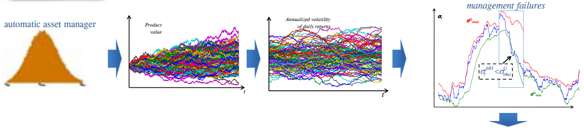


given $n^*=7$

$$\min_{\sigma_2<\sigma_3<\cdots<\sigma_7}\left(\max_{i=2,\ldots,6} mf_i\right)$$

$s.t.\ mf_i - mf_{i-1} \simeq 0$

$with:\ \sigma_2 = 0.25\% \quad \sigma_7 = 25\%$

2nd Pillar

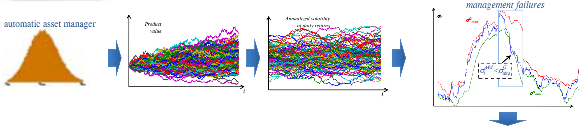
Synthetic risk indicator

Definition of a suitable volatility grid

summarizing:

automatic asset manager

diffusive GARCH to detect *management failures*



given $n^*=7$

$$\min_{\sigma_2<\sigma_3<\cdots<\sigma_7}\left(\max_{i=2,\ldots,6} mf_i\right)$$

$s.t.\ mf_i - mf_{i-1} \simeq 0$

$with:\ \sigma_2 = 0.25\% \quad \sigma_7 = 25\%$

$$\frac{\sigma_2}{\sigma_1}=\frac{\sigma_3}{\sigma_2}=\frac{\sigma_4}{\sigma_3}=\frac{\sigma_5}{\sigma_4}=\cdots=\frac{\sigma_n}{\sigma_{n-1}}=m$$

2nd Pillar

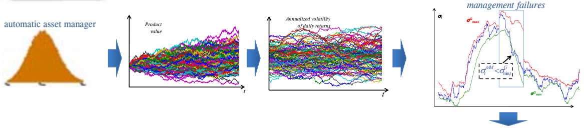
Synthetic risk indicator

Definition of a suitable volatility grid

summarizing:

automatic asset manager

diffusive GARCH to detect *management failures*



given $n^*=7$

$$\min_{\sigma_2<\sigma_3<\cdots<\sigma_7}\left(\max_{i=2,\ldots,6} mf_i\right)$$

$s.t.\ mf_i - mf_{i-1} \simeq 0$

$with:\ \sigma_2 = 0.25\% \quad \sigma_7 = 25\%$

$$\frac{\sigma_2}{\sigma_1}=\frac{\sigma_3}{\sigma_2}=\frac{\sigma_4}{\sigma_3}=\frac{\sigma_5}{\sigma_4}=\cdots=\frac{\sigma_n}{\sigma_{n-1}}=m$$

$$\boxed{m^*=2.5}$$

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
Very Low	0.01%	0.24%
Low	0.25%	0.59%
Medium-Low	0.6%	1.59%
Medium	1.60%	3.99%
Medium-High	4.00%	9.99%
High	10.00%	24.99%
Very High	25.00%	>25.00%

Syllabus

Preliminaries

Three-pillars approach:

1st Pillar: unbundling and performance scenarios

2nd Pillar: the degree of risk

3rd Pillar: recommended investment time horizon

3rd Pillar

The recommended investment horizon

"Risk target" products

"Benchmark" products

"Return target" products

The recommended investment time horizon

for performance target products the recommended minimum investment horizon is inherent to their financial engineering, as the recommended investment horizon is equal to the period of validity (or the time to maturity) of their target

The payoff at maturity uniquely identifies the time when the potential returns are optimized

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3rd Pillar

The recommended investment horizon

"Risk target" products

"Benchmark" products

"Return target" products

The recommended investment time horizon

The use of solutions aimed at ensuring the liquidity and/or marketability of a return target product changes its risk-return profile and its recommended investment time horizon

The event to study from a probabilistic point of view transforms into:

The investment recovers the initial costs and off-sets the running costs at least once

that can be calculated through the concept of

First Passage Time

The "minimum" recommended investment time horizon

3rd Pillar

The recommended investment horizon

"Risk target" products

"Benchmark" products

"Return target" products

The "minimum" recommended investment time horizon

For risk target products, the natural way to define a cost recovery event is also:

The investment recovers the initial costs and off-sets the running costs at least once

that can be calculated through the concept of

First Passage Time

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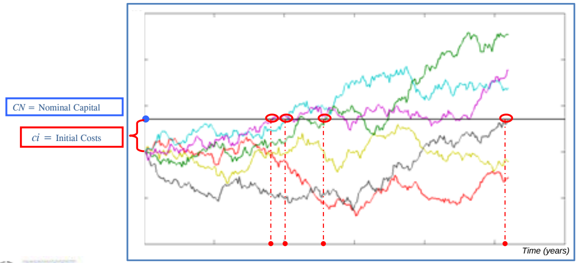
First Passage Time:
First time (expressed in years) such that the value of the Invested Capital (CI) recovers the initial costs and off-sets the running costs.



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The probability of the event:

The investment recovers the initial costs and off-sets the running costs at least once

given a confidence level α, uniquely identifies a time T^* on the cumulative distribution function of the first passage times, i.e.:

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \le T\right] = \alpha \right\}$$

where

$$t^* = \inf\left[t \in \Re^+ : CI_t > CN\right]$$

is the first passage time

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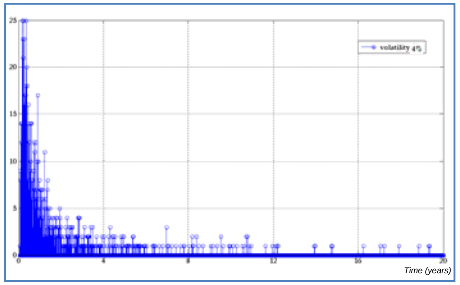
1. Calculation of the probability distribution of the first passage times:



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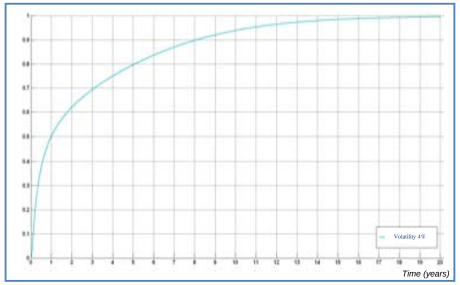
2. Derivation of the cumulative distribution function of the first passage times:



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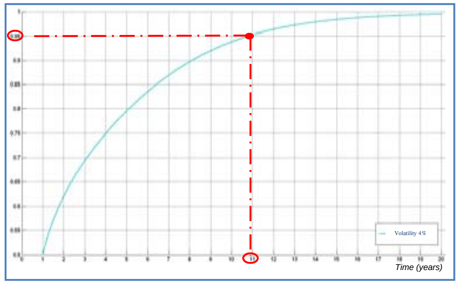
3. The confidence level α uniquely identifies T^* on the cumulative distribution function of the first passage times:



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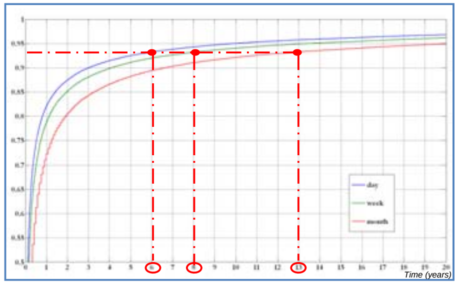
3. The discretization step is relevant in the determination of the cumulative probability function, conditioning the identification of the time horizon, given a fixed level of confidence:



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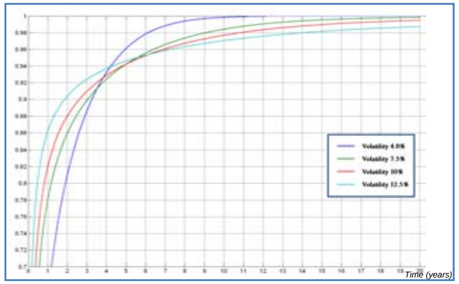
When many probability distribution functions are considered, letting varying volatilities and costs, the problem of correctly identifying a set of minimum thresholds arises:



3rd Pillar

Anyway, the recommended minimum investment time horizon…

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \le T\right] = \alpha \right\}$$

…. Must be coherent with the principle

+ VOLATILITY + TIME HORIZON

The correct way to solve the problem is to set up an operative procedure to select properly each threshold according to the above principle

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3rd Pillar

Connection between probability, volatility and costs

First passage times for the break-even barrier are monitored at infinitesimal time intervals:

$dt \to 0$

$$T^* = \left\{ T \in \Re^+ : \mathrm{P}\left[t^* \le T\right] = \alpha \right\}$$

$$\mathrm{P}\left[t^* \le T\right] = N\left(d_2\left(\frac{CI_0}{CN}\right)\right) + \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \cdot N\left(-d_2\left(\frac{CN}{CI_0}\right)\right)$$

$$d_2(x) = \frac{\log x + \left(\bar{r} - cr - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

Connection between probability, volatility and costs

Asymptotic properties: $T \to \infty$

$\boxed{cr \text{ : recurrent costs as a fixed \%}}$

$\boxed{dt \to 0}$

$$\lim_{T\to\infty} \mathrm{P}\left[t^* \le T\right] = \begin{cases} 1 & if \quad (\bar{r}-cr) \ge \frac{1}{2}\sigma^2 \\ \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} & \boxed{if \quad (\bar{r}-cr) < \frac{1}{2}\sigma^2} \end{cases}$$

Connection between probability, volatility and costs

Under our assumptions:

$\boxed{dt \to 0}$

$$\lim_{T\to\infty} \mathrm{P}\left[t^* \le T\right] = \begin{cases} 1 & if \quad (\bar{r}-cr) \ge \frac{1}{2}\sigma^2 \\ \left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} & \boxed{if \quad (\bar{r}-cr) < \frac{1}{2}\sigma^2} \end{cases}$$

For a given level of costs, it is possible to analytically derive the connection between volatility and time horizon

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Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

FIRST ORDER SENSITIVITY ANALYSIS

$$\boxed{\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1}\right)}$$

⬇

FIRST ORDER ASYMPTOTIC CONDITION

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1}\right)$$

1. $(\bar{r}-cr) > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $(\bar{r}-cr) \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0$

The existence of two alternative states of nature requires to verify whether both of them make sense in financial terms under the risk-neutral measure.

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$\boxed{cr = 0}$

1. $\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0$

Being running costs a specific feature of any financial product they would interfere with the task of identifying which of the two conditions has a sound financial meaning. Therefore, they will be temporarily neglected.

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$\boxed{cr = 0}$

1. $\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0$
2. $\cancel{\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0}$

Since it is safe to assume a positive interest rate r in financial markets, only condition 1. correctly captures the connection between volatility and time horizon.

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$\boxed{cr = 0}$

1. $\boxed{\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0}$
2. $\cancel{\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0}$

As $T \to \infty$ condition 1. implies that the cumulative distribution function P is a strictly decreasing function of the volatility, i.e.:

$$\boxed{\forall \sigma_i, \sigma_j \in \Re^+, \sigma_j > \sigma_i \Rightarrow \mathrm{P}(\sigma_j) < \mathrm{P}(\sigma_i)}$$

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$\boxed{cr = 0}$

1. $\boxed{\bar{r} > 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} < 0}$
2. $\cancel{\bar{r} \le 0 \Leftrightarrow \frac{d\mathrm{P}}{d\sigma} \ge 0}$

In other words, for a given a confidence level, as the volatility grows, the recommended investment time horizon increases as well:

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

Connection between probability, volatility and costs

$T \to \infty, dt \to 0$

$$\frac{d\mathrm{P}}{d\sigma} = \left(-4\frac{\bar{r}}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2\bar{r}}{\sigma^2}-1}\right)$$

$\boxed{cr = 0}$

1. $\bar{r} > 0 \Leftrightarrow \boxed{\frac{d\mathrm{P}}{d\sigma} < 0}$
2. $\bar{r} \le 0 \Leftrightarrow \boxed{\frac{d\mathrm{P}}{d\sigma} \ge 0}$

$$\boxed{\exists T^* \in [0, \infty[: \frac{d\mathrm{P}}{d\sigma} = 0}$$

Furthermore, condition 1. alone is sufficient to guarantee a minimum time T^* beyond which the following strong condition holds:

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

$$\frac{dP}{d\sigma} = \left(-4\frac{(\bar{r}-cr)}{\sigma^3}\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \right)$$

$$cr > 0$$

1. $(\bar{r}-cr) > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0$

2. $\cancel{(\bar{r}-cr) \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0}$

$$\exists T^* \in [0,\infty[: \frac{dP}{d\sigma} = 0$$

Generalizing…

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

$$\frac{d^2P}{d\sigma^2} = \frac{4}{\sigma^4}(\bar{r}-cr)\ln\left(\frac{CN}{CI_0}\right)\left(\frac{CN}{CI_0}\right)^{\frac{2(\bar{r}-cr)}{\sigma^2}-1} \cdot \left[1+\frac{4(\bar{r}-cr)}{\sigma^2}\ln\left(\frac{CN}{CI_0}\right)\right]$$

$$(\bar{r}-cr) > 0 \Rightarrow \frac{d^2P}{d\sigma^2} > 0$$

SECOND ORDER
ASYMPTOTIC CONDITION

Second Order
Sensitivity
Analysis

Connection between probability, volatility and costs

$$T \to \infty, dt \to 0$$

1.
$$\begin{cases} (\bar{r}-cr) > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0 \\ (\bar{r}-cr) > 0 \Rightarrow \frac{d^2P}{d\sigma^2} > 0 \end{cases}$$

2. $\cancel{(\bar{r}-cr) \le 0 \Leftrightarrow \frac{dP}{d\sigma} \ge 0}$

$$\exists T^* \in [0,\infty[: \frac{dP}{d\sigma} = 0$$

Summarizing the results of the asymptotic analysis in continuous time:

- As T →∞, for given a confidence level, more volatility implies a larger recommended investment time horizon
- It is always possible to find a <u>minimum</u> and finite time T^*, beyond which the strong condition

+VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

holds

DETERMINATION OF THE INVESTMENT TIME HORIZON

General Framework:

$$\begin{cases} T \to \infty \\ dt \to 0 \\ P(\infty,\sigma) \\ 1. \begin{cases} (\bar{r}-cr) > 0 \Leftrightarrow \frac{dP}{d\sigma} < 0 \\ (\bar{r}-cr) > 0 \Rightarrow \frac{d^2P}{d\sigma^2} > 0 \end{cases} \end{cases} \Rightarrow \begin{cases} \boxed{T \ \ finite} \\ dt \to 0 \\ P(T,\sigma) \\ (\bar{r}-cr) > 0 \Leftrightarrow \lim_{T\to\infty}\frac{\partial P(T,\sigma)}{\partial\sigma} < 0 \\ (\bar{r}-cr) > 0 \Rightarrow \lim_{T\to\infty}\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2} > 0 \end{cases}$$

In order to determine effectively the investment time horizon, it is necessary to abandon the asymptotic environment and to shift the analysis of condition 1. in a finite time framework.

DETERMINATION OF THE INVESTMENT TIME HORIZON

FIRST ORDER SENSITIVITY ANALYSIS

$$\frac{\partial P(T,\sigma)}{\partial\sigma}$$

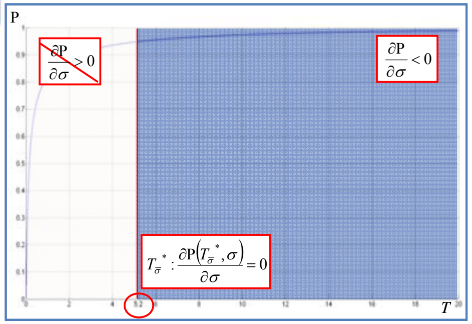
At a finite time T, the sufficient condition of the first order that allows to state the core relationship

+ volatility + time horizon

is then specified in the following form:

$$\begin{cases} \cancel{\left.\frac{\partial P(T,\sigma)}{\partial\sigma}\right|_{\sigma=\bar{\sigma}} > 0 \quad if \quad 0 \le T < T^*_{\bar{\sigma}}} \\ \left.\frac{\partial P(T,\sigma)}{\partial\sigma}\right|_{\sigma=\bar{\sigma}} \le 0 \quad if \quad T \ge T^*_{\bar{\sigma}} \end{cases}$$

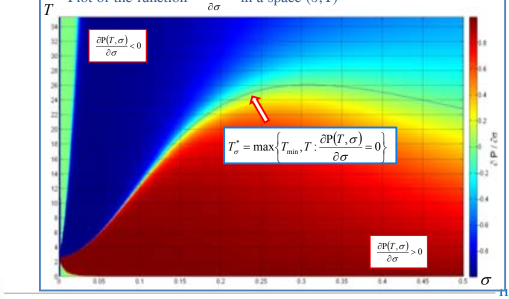
DETERMINATION OF THE INVESTMENT TIME HORIZON



DETERMINATION OF THE INVESTMENT TIME HORIZON

FIRST ORDER SENSITIVITY ANALYSIS

Plot of the function $\frac{\partial P(T,\sigma)}{\partial\sigma}$ in a space (σ,T)



DETERMINATION OF THE INVESTMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS

$$\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$$

Given the monotonicity condition of the probability distribution with respect to volatility, i.e.:

$$\forall\sigma_i,\sigma_j \in \Re^+, \sigma_j > \sigma_i \Rightarrow P(\infty,\sigma_j) < P(\infty,\sigma_i)$$

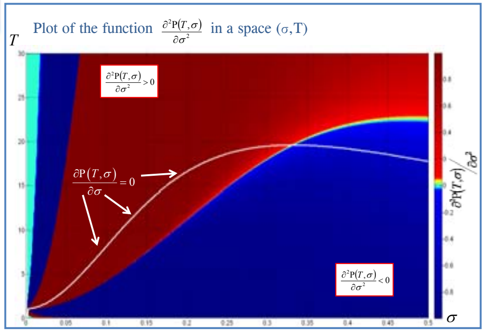
In order to fulfill this condition, it's necessary to restrict the analysis in the region where the probability function is strictly increasing, i.e.:

$$\left.\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}\right|_{T=T_\sigma^*} > 0 \Rightarrow T_\sigma^* \text{ increasing}$$

$$\cancel{\left.\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}\right|_{T=T_\sigma^*} < 0 \Rightarrow T_\sigma^* \text{ decreasing}}$$

DETERMINATION OF THE INVESTMENT TIME HORIZON

Plot of the function $\frac{\partial^2 P(T,\sigma)}{\partial\sigma^2}$ in a space (σ,T)



DETERMINATION OF THE INVESTMENT TIME HORIZON

SECOND ORDER SENSITIVITY ANALYSIS

$$\frac{\partial^2 \mathrm{P}(T,\sigma)}{\partial\sigma^2}$$

Having defined the maximum time in the form:

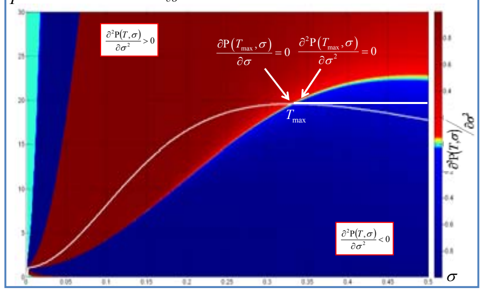
$$\begin{cases}\sigma \in \Re^+ \\ T_{\max} \in T_\sigma^*\end{cases} : \frac{\partial^2 \mathrm{P}(T_{\max},\sigma)}{\partial\sigma^2} = 0$$

The sufficient condition of the 2° order is specified as:

$$T^* = \begin{cases} T_\sigma^* \; if \; \left.\dfrac{\partial^2 \mathrm{P}(T,\sigma)}{\partial\sigma^2}\right|_{T=T_\sigma^*} \geq 0 \\ T_{\max} \; if \; \left.\dfrac{\partial^2 \mathrm{P}(T,\sigma)}{\partial\sigma^2}\right|_{T=T_\sigma^*} < 0 \end{cases}$$

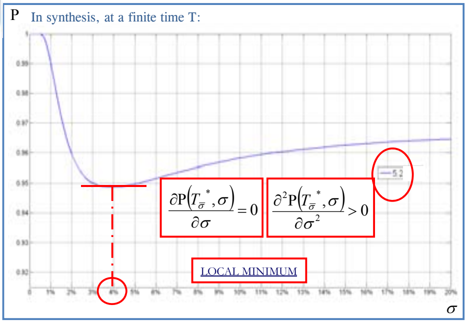
DETERMINATION OF THE INVESTMENT TIME HORIZON

Plot of the function $\frac{\partial^2 \mathrm{P}(T,\sigma)}{\partial\sigma^2}$ in a space (σ,T)



DETERMINATION OF THE INVESTMENT TIME HORIZON

In synthesis, at a finite time T:



+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

STRONG CONVERGENCE LEMMA for times

Given a sequence of financial products F_j with volatility σ_j and recalling the first order sufficient condition:

$$T_\sigma^* = \max\left\{T_{\min}, T : \frac{\partial \mathrm{P}(T,\sigma)}{\partial\sigma} = 0\right\}, \quad \forall\sigma \in \Re^+$$

the first order sufficient condition can be specified for the class of products F_j in the following form:

$$T_{\sigma_j}^{\varepsilon_j} : \mathrm{P}\left(T_{\sigma_j}^{\varepsilon_j}, \sigma_{j+1}\right) = \mathrm{P}\left(T_{\sigma_j}^{\varepsilon_j}, \sigma_j\right)$$

It therefore holds the following strong convergence relation with respect to times:

$$\lim_{\sigma_{j+1}\to\sigma_j} T_{\sigma_j}^{\varepsilon_j} = T_{\sigma_j}^*$$

where $\varepsilon_j = (\sigma_{j+1} - \sigma_j) > 0.$

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In order to have an intuitive explanation of the lemma, let's consider the following volatility levels:

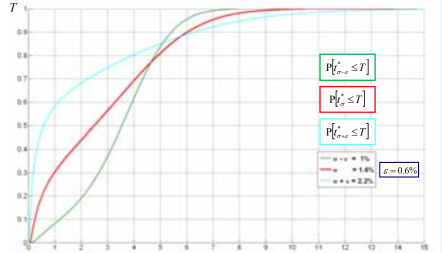
$\sigma - \varepsilon, \varepsilon \in \Re^+$ | σ | $\sigma + \varepsilon, \varepsilon \in \Re^+$

and the respective probability distribution functions, i.e.:

$\mathrm{P}[t_{\sigma-\varepsilon}^* \leq T]$ | $\mathrm{P}[t_\sigma^* \leq T]$ | $\mathrm{P}[t_{\sigma+\varepsilon}^* \leq T]$

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
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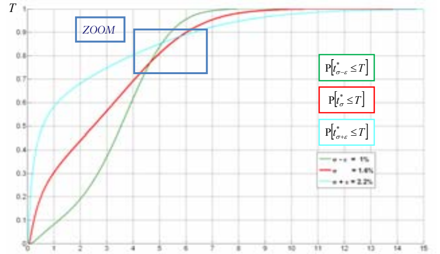

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
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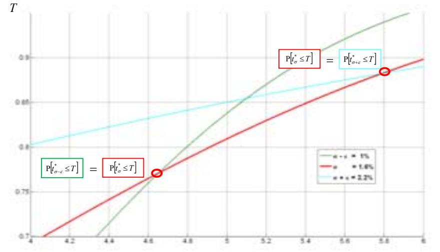

+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
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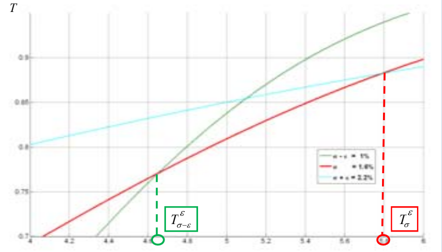
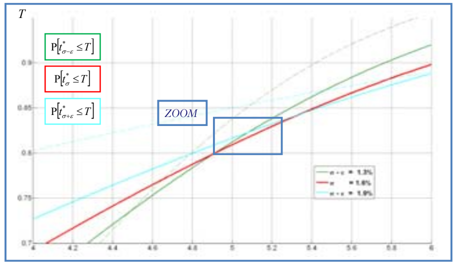


3rd Pillar

\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons



3rd Pillar

\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
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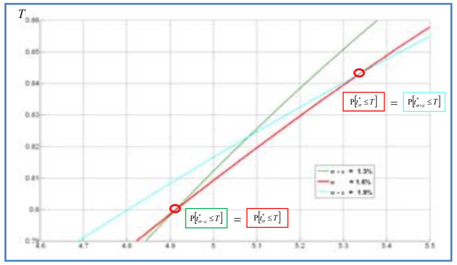

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\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
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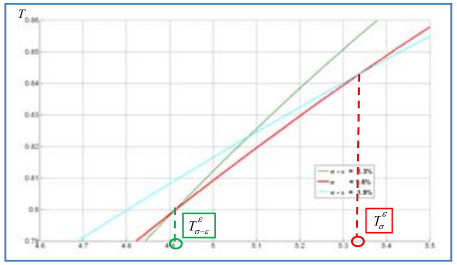

3rd Pillar

\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
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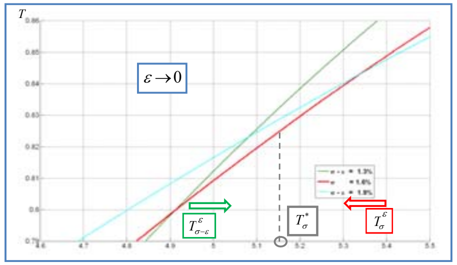
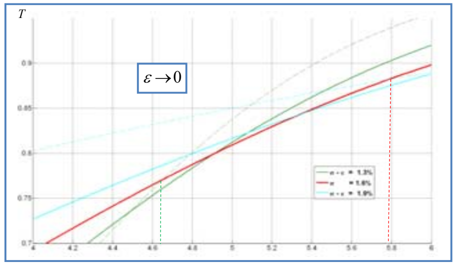


3rd Pillar

\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In synthesis..

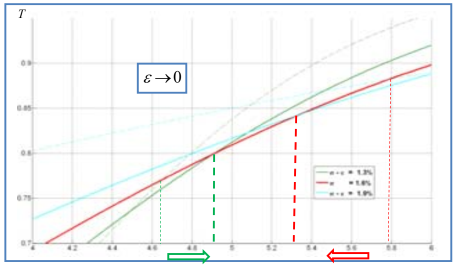


3rd Pillar

\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In synthesis..



3rd Pillar

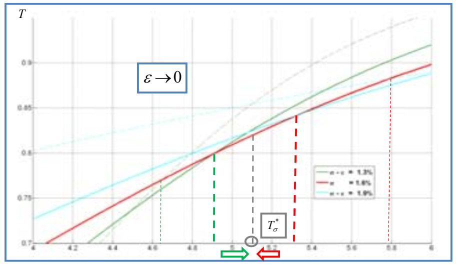
\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

In synthesis..

$\sigma = 1.6\%$

$\boxed{T^*_{\sigma} = 5.1}$ years

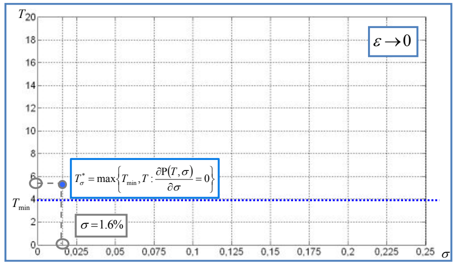


3rd Pillar

\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

The time is characterized on the curve of minimum times..



$$T^*_{\sigma} = \max\left\{T_{\min}, T : \frac{\partial \mathrm{P}(T,\sigma)}{\partial \sigma} = 0\right\}$$

3rd Pillar

\+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

Generalizing the lemma <u>for all σ,</u> the following characterization of the first order sufficient condition is given:

$\sigma-\varepsilon, \varepsilon \in \Re^+$	σ	$\sigma+\varepsilon, \varepsilon \in \Re^+$
$\mathrm{P}\left[t^*_{\sigma-\varepsilon} \le T\right]$	$\mathrm{P}\left[t^*_{\sigma} \le T\right]$	$\mathrm{P}\left[t^*_{\sigma+\varepsilon} \le T\right]$

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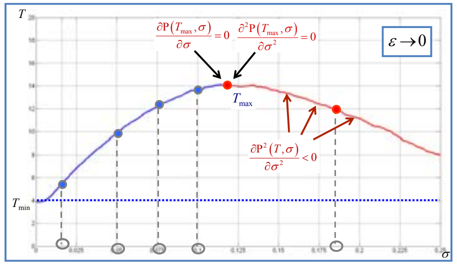
+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

The time is characterized on the curve of minimum times..



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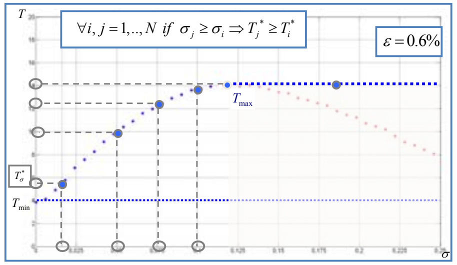
SECOND ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

Weak monotonicity condition of times w.r.t. volatility



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+ VOLATILITY + RECOMMENDED INVESTMENT TIME HORIZON

FIRST ORDER SUFFICIENT CONDITION
to determine a sequence of consistent time horizons

Formally, for any sequence of products with volatility σ_j, defined in a given class of costs (ci,cr):

Strong convergence lemma for times
First order sufficient condition

Weak monotonicity condition of times w.r.t. volatility
Second order sufficient condition

$$\forall j=1,...N,\sigma_{j+1}>\sigma_j,$$
$$T_{j+1}^*=\max\left\{\ T_j^*,\ T\in[T_{\min},T_{\max}]:\mathrm{P}\left[t_{\sigma_{j+1}}^*\leq T\right]=\mathrm{P}\left[t_{\sigma_j}^*\leq T\right]=\alpha_{j+1}^*\right\}$$

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Integrated risk measurement and representation for non-equity products: how to frame material risks over the time horizon of the investment

Marcello Minenna – Head of Quantitative Analysis Unit, Consob

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