1 Pricing Formulae for General Models

1.1 Quasi BSM Formula

The classic Black–Scholes–Merton style formula was first developed by Heston (1993) and then used by Bates (1996), Bakshi *et al* (1997), Duffie *et al* (2000) and Lewis (2001) for Levy processes. Sepp (2004) proved that this formula is still valid for general processes.

PROPOSITION 1 *Assuming that a characteristic function* $\tilde{f}_k(\xi) = E[e^{i\xi \ln S_t}]$, *for* $k = 1, 2$, *corresponding to each model analysed, exists in an analytical form, there exists the following representation for the cumulative probability function* P_k:

$$\Pi_k[\ln S_T \geq \ln[K] \mid \ln S_t] = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\left[\frac{e^{-i\xi \ln[K]}}{i\xi}\tilde{f}_k(\xi)\right] d\xi \qquad (1.1)$$

THEOREM 1 *Given the binary variable* $\varphi = 1$ *for a call and* $\varphi = -1$ *for a put, then the value of a European contingent claim* $F(S_t, t)$ *that pays* $\max[\varphi(S_T - K), 0]$ *at terminal time* T *has the form:*

$$F(S_t, t) = \varphi[S_t P_1(\varphi) - e^{-r(T-t)} P_2(\varphi)] \qquad (1.2)$$

where:

$$P_k(\varphi) = \frac{1-\varphi}{2} + \varphi \Pi_k$$

This formula has been highly criticised by several recent papers in the field of Fourier analysis and the results are so clear that this approach can be considered relatively obsolete with respect to the most robust single integration approach (see below). A detailed survey on this topic can be found in Lewis (2001), but also the Sepp (2004) and Lee (2004) papers provide useful hints. Nevertheless, this classic method is still used by practitioners.

1.2 The Single Integration Formula

The quasi BS formula has a nice representation in terms of the density probability function associated with a well-defined characteristic function. In order to derive that, let us prove the following preliminary result.

PROPOSITION 2 *Equation (1.1) can be expressed in the following alternative form:*

$$P_k(\ln S_t) = \int_{\ln K}^{+\infty} q_k(\ln S_T \mid \ln S_t)\, d\ln S_T \qquad (1.3)$$

where q_k is the conditioned density probability function of the logarithm of the underlying financial instrument.

PROOF Recalling Expression (1.1) and without loss of generality setting $\varphi = 1$:

$$P_k = \Pi_k[\ln S_T \geq \ln[K] \mid \ln S_t] \tag{1.4}$$

Equation (1.4) has the alternative representation:

$$P_k = E_k(1_{[\ln S_T \geq \ln K]} \mid \ln S_t) = \int_{-\infty}^{+\infty} 1_{[\ln S_T \geq \ln K]} f_k(\ln S_T \mid \ln S_t)\, \mathrm{d} \ln S_T$$

It follows that:[1]

$$P_k = \int_{\ln K}^{+\infty} f_k(\ln S_T \mid \ln S_t)\, \mathrm{d} \ln S_T \tag{1.3'}$$

□

THEOREM 2 *Let $F_T(\ln K)$ be the desired value of a T maturity European option with strike $\mathrm{e}^{\ln K}$. The value of a European contingent claim $F(S_t, t)$ that pays $\max[\varphi(S_T - K), 0]$ at terminal time T:*

$$F(S_t, t) = \varphi[S_t P_1(\varphi) - \mathrm{e}^{-r(T-t)} P_2(\varphi)] \tag{1.5}$$

can be expressed in terms of the risk-neutral density probability function $q(\ln S_T)$ associated with the characteristic function $\bar{f}_k(\xi)$ as follows:

$$F(S_t, t) = \varphi\left[\mathrm{e}^{-r(T-t)} \int_{\ln K}^{\infty} (\mathrm{e}^{\ln S_{T_t}} - \mathrm{e}^{\ln K}) q_2(\ln S_T)\, \mathrm{d} \ln S_T\right] \tag{1.6}$$

PROOF Let us consider, without loss of generality, $\varphi = 1$. Expression (1.2) assumes the form:

$$C_t = S_t P_1 - K\, \mathrm{e}^{-r(T-t)} P_2 \tag{1.2'}$$

Substituting the result (1.3) into Equation (1.2'):

$$C_t = S_t \int_{\ln K}^{+\infty} q_1(\ln S_T \mid \ln S_t)\, \mathrm{d} \ln S_T - K\, \mathrm{e}^{-r(T-t)} \int_{\ln K}^{+\infty} q_2(\ln S_T \mid \ln S_t)\, \mathrm{d} \ln S_T$$

It follows that:[2]

$$\begin{aligned} C_t = {} & S_t \int_{\ln K}^{+\infty} \Lambda_{1,2} q_2(\ln S_T \mid \ln S_t)\, \mathrm{d} \ln S_T \\ & - K\, \mathrm{e}^{-r(T-t)} \int_{\ln K}^{+\infty} q_2(\ln S_T \mid \ln S_t)\, \mathrm{d} \ln S_T \end{aligned} \tag{1.7}$$

Ignoring the subscript of q and with a little algebra:

$$C_t = \mathrm{e}^{-r(T-t)} \int_{\ln K}^{+\infty} [\mathrm{e}^{r(T-t)} S_t \Lambda_{1,2} - \mathrm{e}^{\ln K}] q(\ln S_T \mid \ln S_t)\, \mathrm{d} \ln S_T \tag{1.8}$$

The hypothesis of risk neutrality is now used to define q properly. In order to do so, a measure theory result is used:[3]

$$E\left[\frac{S_T}{\mathrm{e}^{rT}} \,\middle|\, \frac{S_t}{\mathrm{e}^{rt}}\right] = \frac{S_t}{\mathrm{e}^{rt}} \tag{1.9}$$

Expression (1.9) can be rearranged as:

$$E\left[S_T \,\middle|\, \frac{S_t}{\mathrm{e}^{rt}}\right] = \mathrm{e}^{r(T-t)} S_t \tag{1.10}$$

Using the above condition in the definition of Radon–Nykodim derivative $\Lambda_{1,2}$ specifies q completely under the risk-neutral measure.

Hence, let us specify $\Lambda_{1,2}$ as follows:

$$\Lambda_{1,2} = \frac{S_T}{E\left[S_T \mid \frac{S_t}{e^{rt}}\right]}$$

So:

$$E\left[S_T \,\middle|\, \frac{S_t}{e^{rt}}\right] = \frac{S_T}{\Lambda_{1,2}} \tag{1.11}$$

Substituting Equation (1.11) into (1.10):

$$\Lambda_{1,2} = \frac{S_T}{e^{r(T-t)}S_t} \tag{1.12}$$

The quantity expressed in Expression (1.12) is unique[4] and it allows us to specify q under the risk-neutral measure. In fact:

$$C_t = e^{-r(T-t)} \int_{\ln K}^{+\infty} [e^{\ln S_T} - e^{\ln K}] q(\ln S_T \mid \ln S_t) \, d \ln S_T \tag{1.6'}$$

□

1.2.1. The Carr–Madan Representation

The original representation of the single integration formula by Carr and Madan (1999) is outlined here. This approach has been refined and improved by the work of Lewis (2001) and Lee (2004).

PROPOSITION 3 *Let* $C_T(\ln K)$ *be the desired value of a* T *maturity call option with strike* $e^{\ln K}$. *The risk-neutral density of the log price* $\ln S_T$ *is denoted as* $q_T(s)$. *Then let:*

$$\phi_T(\xi) = \int_{-\infty}^{\infty} e^{i\xi s} q_T(\ln S_T) \, d \ln S_T \tag{1.13}$$

be the characteristic function (or Fourier transform) of this density.

The call price can be expressed in the following form:

$$C_t(\ln K) = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{\infty} \Re\left[e^{-iv\ln K} \frac{e^{-r(T-t)}\phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}\right] dv \tag{1.14}$$

where $\alpha \in [0, \infty[$ *is known as the damping parameter.*

PROOF See Carr and Madan (1999). □

Obviously, this formula has the advantage to invert only one characteristic function, instead of computing two different inversions, as in the canonical approach. The advantage is both in terms of computational time and accuracy since also quadrature flaws are doubled in the classical case. As an added benefit the denominator of the integrand is now a quadratic function in the integrating variable v, and as such decays faster than the integrands in Equation (1.1). Finally, the Carr–Madan representation (1.14) allows us to split the problem of tiny option prices from the problem of machine size precision since $e^{-\alpha \ln K}$ serves as a scaling factor. An appropriate choice of α enables us to find a scaling value, which allows us to calculate arbitrarily small option prices. A naive but reasonable approach to the problem of the correct choice of α is presented in Section 5.

In Lewis (2001), a direct mapping of the log spot price characteristic function from the Fourier space is also used, but from a more general point of view. In fact, in the work of Lewis, a contour integral in the complex plane is used to give an alternative representation to Equation (1.14) is represented as a contour integral in the complex plane. The problem of the impact of the damping parameter α is then characterised as the effect on the price of the choice of a particular strip of integration in the complex plane, giving an intuitive insight about the instability issue. The improved version of Lee (2004) of the single integration formula shows some other decisive features in accuracy, but we choose to test this somewhat old dated formula as it is the most used alternative to the quasi Black–Scholes–Merton formula among practitioners.

Notes

1 See Minenna (2006), Part I, Chapter 6.
2 See Minenna (2006), Part II, Chapter 10.
3 See Minenna (2006), Part II, Chapter 10.
4 See Minenna (2006), Part II, Chapter 10.