15.2.7.3.2. The shift into the à-la BSM pricing context The Cauchy problem as in (15.463) is now specified in the classic à-la BSM form, ie,

$$\begin{aligned} C_t(S, v, t, T) &= S_t P_1(S, v, t, T) - Ke^{-r(T-t)} P_2(S, v, t, T) \quad \text{or} \\ C_t(x, v, t) &= e^{x_t} P_1(x, v, \tau) - Ke^{-r(T-t)} P_2(x, v, t) \end{aligned} \tag{15.464}$$

where P_1, P_2 are probability measures, or P_j for $k = 1, 2$.

Equation (15.464) is used to compute the partial derivatives in order to make (15.463) explicit:

$$\frac{\partial C}{\partial x} = e^x P_1(x, v, t) + e^x \frac{\partial P_1}{\partial x} - Ke^{-r(T-t)} \frac{\partial P_2}{\partial x} \tag{15.465}$$

$$\frac{\partial^2 C}{\partial x^2} = e^x P_1(x, v, t) + 2e^x \frac{\partial P_1}{\partial x} + e^x \frac{\partial^2 P_1}{\partial x^2} - Ke^{-r(T-t)} \frac{\partial^2 P_2}{\partial x^2} \tag{15.466}$$

$$\frac{\partial C}{\partial t} = e^x \frac{\partial P_1}{\partial t} - K\left[e^{-r(T-t)} \frac{\partial P_2}{\partial t} + re^{-r(T-t)} P_2(x, v, t)\right] \tag{15.467}$$

$$\frac{\partial C}{\partial v} = e^x \frac{\partial P_1}{\partial v} - Ke^{-r(T-t)} \frac{\partial P_2}{\partial v} \tag{15.468}$$

$$\frac{\partial^2 C}{\partial v^2} = e^x \frac{\partial^2 P_1}{\partial v^2} - Ke^{-r(T-t)} \frac{\partial^2 P_2}{\partial v^2} \tag{15.469}$$

$$\frac{\partial^2 C}{\partial x \partial v} = e^x \frac{\partial P_1}{\partial v} + e^x \frac{\partial^2 P_1}{\partial x \partial v} - Ke^{-r(T-t)} \frac{\partial^2 P_2}{\partial x \partial v} \tag{15.470}$$

The expected value of (15.461) based on (15.464) remains to be expanded:

$$E\left\{C\Big[\underbrace{x+\ln(J+1)}_{\text{argument } x \text{ of } (15.464)},v,t\Big]-C[x,v,t]\right\}$$

factorising P_1 and P_2, one obtains

$$\begin{aligned}&E\{C[x+\ln(J+1),t]-C[x,t]\}\\&\quad=E\{e^x[[(J+1)P_1[x+\ln(J+1),v,t]-P_1(x,v,t)]\\&\qquad-Ke^{-r(T-t)}[P_2[x+\ln(J+1),v,t]-P_2(x,v,t)]]\}\end{aligned}\tag{15.471}$$

Then (15.466), (15.467), (15.468), (15.469), (15.470) and (15.471) are substituted in (15.463), to obtain

$$\begin{aligned}&e^x\Bigg[-\lambda\mu P_1(x,v,t)+\frac{\partial P_1}{\partial t}+\frac{\partial P_1}{\partial x}\left(r-\lambda\mu+\frac{1}{2}v\right)+\frac{1}{2}\left(\frac{\partial^2P_1}{\partial v^2}(\sigma^2v)\right)+\frac{\partial^2P_1}{\partial x\partial v}(v\sigma\rho)\\&\quad+\frac{1}{2}\left(\frac{\partial^2P_1}{\partial x^2}v\right)+\frac{\partial P_1}{\partial v}[\kappa(\theta-v)-\widetilde{\chi}v+v\sigma\rho]\\&\quad+\lambda E[(J+1)P_1[x+\ln(J+1),v,t]-P_1(x,v,t)]\Bigg]\\&\quad-Ke^{-r(T-t)}\Bigg[\frac{\partial P_2}{\partial t}+\frac{\partial P_2}{\partial x}\left(r-\lambda\mu-\frac{1}{2}v\right)+\frac{1}{2}\frac{\partial^2P_2}{\partial v^2}(\sigma^2v)+\frac{\partial^2P_2}{\partial x\partial v}(v\sigma\rho)\\&\quad+\frac{1}{2}v(\frac{\partial^2P_2}{\partial x^2})+\frac{\partial P_2}{\partial v}[\kappa(\theta-v)-\widetilde{\chi}v]+\lambda E[P_2[x+\ln(J+1),v,t]\\&\quad-P_2(x,v,t)]\Bigg]=0\end{aligned}$$

Since $e^x>0$ and $K>0$, in order to verify the equivalence above, it is enough that

$$\begin{cases}-\lambda\mu P_1(x,v,t)+\dfrac{\partial P_1}{\partial t}+[r-\lambda\mu]\dfrac{\partial P_1}{\partial x}+\dfrac{1}{2}v\dfrac{\partial P_1}{\partial x}+\dfrac{1}{2}\dfrac{\partial^2P_1}{\partial v^2}(\sigma^2v)\\\quad+\dfrac{\partial^2P_1}{\partial x\partial v}(v\sigma\rho)+\dfrac{1}{2}\dfrac{\partial^2P_1}{\partial x^2}v+\dfrac{\partial P_1}{\partial v}[\kappa\theta-v(\kappa+\widetilde{\chi}-\sigma\rho)]\\\quad+\lambda E[(J+1)P_1[x+\ln(J+1),v,t]-P_1(x,v,t)]=0\\\dfrac{\partial P_2}{\partial t}+[r-\lambda\mu]\dfrac{\partial P_2}{\partial x}-\dfrac{1}{2}v\dfrac{\partial P_2}{\partial x}+\dfrac{1}{2}\dfrac{\partial^2P_2}{\partial v^2}(\sigma^2v)+\dfrac{\partial^2P_2}{\partial x\partial v}(v\sigma\rho)+\dfrac{1}{2}\dfrac{\partial^2P_2}{\partial x^2}v\\\quad+\dfrac{\partial P_2}{\partial v}[\kappa\theta-v(\kappa+\widetilde{\chi})]+\lambda E[P_2[x+\ln(J+1),v,t]-P_2(x,v,t)]=0\end{cases}\tag{15.472}$$

Equations (15.472) are equivalent forms of PDE (15.463) in the à-la BSM call pricing context as in (15.464). To identify the Cauchy problems in (15.472), equivalent to problem (15.463) and then to (15.456), one has to derive the boundary condition; for this purpose, the characteristics of function P_j are specified at time $t = T$ in order to determine, based on (15.472), the boundary condition under (15.463a) of the PDE (15.463), ie,

$$P_j(x_T, v_T, T) = \begin{cases} 1 & \text{if } (e^{x_T} - K) \geq 0 \\ 0 & \text{if } (e^{x_T} - K) < 0 \end{cases} \quad \text{for } j = 1, 2$$

by applying the logarithm

$$P_j(x_T, v_T, T) = \begin{cases} 1 & \text{if } x_T \geq \ln K \\ 0 & \text{if } xT < \ln K \end{cases} \quad \text{for } j = 1, 2$$

and using the definition of the index function (see Definition 6.3), one obtains

$$P_j(x_T, v_T, T) = 1_{(x_T \geq \ln K)}$$

The following equations are the transformed Cauchy problem in the à-la BSM call pricing environment determined as in (15.463) and its boundary conditions:

$$\underbrace{\frac{\partial P_1}{\partial t} + \left[r + \frac{1}{2}v\right]\frac{\partial P_1}{\partial x} + \frac{1}{2}\frac{\partial^2 P_1}{\partial v^2}(\sigma^2 v) + \frac{\partial^2 P_1}{\partial x \partial v}(v\sigma\rho) + \frac{1}{2}\frac{\partial^2 P_1}{\partial x^2}v}_{\text{Deterministic component}}$$
$$+ \underbrace{\frac{\partial P_1}{\partial v}[\kappa\theta - v(\kappa + \widetilde{\chi} - \sigma\rho)]}_{\text{Deterministic component}}$$
$$\underbrace{- \lambda\mu P_1(x, v, t) + \lambda E[(J+1)P_1[x + \ln(J+1), v, t] - P_1(x, v, t)] - \lambda\mu\frac{\partial P_1}{\partial x}}_{\text{Pure jump component}} = 0 \tag{15.473}$$

$$P_1(x_T, v_T, T) = 1_{(x_T \geq \ln K)} \tag{15.474}$$

$$\underbrace{\frac{\partial P_2}{\partial t} + \left[r - \frac{1}{2}v\right]\frac{\partial P_2}{\partial x} + \frac{1}{2}\frac{\partial^2 P_2}{\partial v^2}(\sigma^2 v) + \frac{\partial^2 P_2}{\partial x \partial v}(v\sigma\rho) + \frac{1}{2}\frac{\partial^2 P_2}{\partial x^2}v}_{\text{Deterministic component}}$$

$$+ \underbrace{\frac{\partial P_2}{\partial v}[\kappa\theta - v(\kappa + \widetilde{\chi})]}_{\text{Deterministic component}}$$

$$+ \underbrace{\lambda[P_2[x + \ln(J+1), v, t] - P_2(x, v, t)] - \lambda\mu\frac{\partial P_2}{\partial x}}_{\text{Pure jump component}} = 0 \tag{15.475}$$

$$P_2(x_T, v_T, T) = 1_{(x_T \geq \ln K)} \tag{15.476}$$

Then the characteristics of the probability measure P_j at time t remain to be determined. For this purpose, (15.473) and (15.475) are interpreted by using the deterministic components of the Feynman–Kac formula (see Theorem 12.21). Actually, the corresponding SDEs of (15.473) and (15.475) may be determined, ie,

$$dx_t^{(1)} = (r + \tfrac{1}{2}v_t)\,dt + \sigma\sqrt{v_t}\,dz_t^{(1)} \quad \text{with } x_t = x \tag{15.477}$$

$$dv_t^{(1)} = (\kappa\theta - (\kappa + \widetilde{\chi} - \rho\sigma)v_t)\,dt + \sigma\sqrt{v_t}\,dz_t^{(2)} \quad \text{with } v_t = v \tag{15.478}$$

$$dx_t^{(2)} = (r - \tfrac{1}{2}v_t)\,dt + \sigma\sqrt{v_t}\,dz_t^{(1)} \quad \text{with } x_t = x \tag{15.479}$$

$$dv_t^{(2)} = (\kappa\theta - (\kappa + \widetilde{\chi})v_t)dt + \sigma\sqrt{v_t}\,dz_t^{(2)} \quad \text{with } v_t = v \tag{15.480}$$

with $dz_t^{(1)} \cdot dz_t^{(2)} = \rho\,dt$. Then (12.103) is rearranged, ie,

$$P_j(x_t, v_t, t) = E_j[1_{(x_T \geq \ln K)} | x_t = x, v_t = v]$$

by simplifying, one has

$$P_j(x_t, v_t, t) = P_j[x_T \geq \ln K | x_t = x, v_t = v]$$

for $j = 1$ and 2, respectively.

By denoting the equivalence $x_t = x$ by x_t and the equivalence $v_t = v$ by v_t, one obtains the characteristics of the probability measure P_j at a generic time t:

$$P_j(x_t, v_t, t) = P_j(x_T \geq \ln K \mid x_t, v_t) \tag{15.481}$$