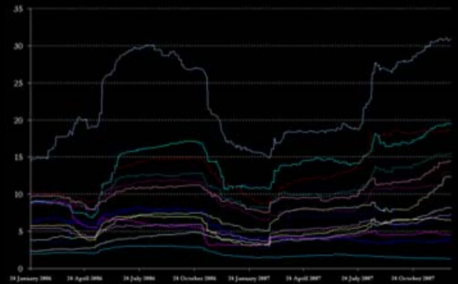
AN APPLICATION OF THE GARCH DIFFUSIVE APPROACH TO THE DEVELOPMENT OF VOLATILITY MEASURES ON THE RISK PROFILE OF MUTUAL FUNDS



XXXII CONVEGNO AMASES – TRENTO, 1-4 SETTEMBRE 2008

MARCELLO MINENNA – GIOVANNA M. BOI

CONSOB

Syllabus

- Volatility
- The GARCH Diffusive Approach
- Application to Flexible Funds Risk Assessment
- Empirical Evidence on the Risk Profile of Flexible Mutual Funds
- Conclusions

CONSOB

Syllabus

- Volatility
  - Importance and Relationship with other Risk Measures
  - Random Variable and Stochastic Process

CONSOB

Volatility is usefully employed in several problems of mathematical finance, such as:

Derivatives Pricing

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Mutual Funds Risk Assessment

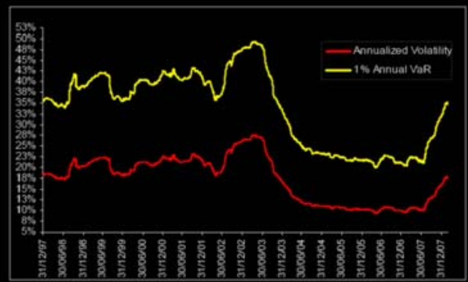
$$d\ln NAV_t = b(t, \ln NAV_t)dt + \sigma(t, \ln NAV_t)dW_t$$

Term Structure Modelling

$$df_t = \alpha(t,T)dt + \sigma(t,T)dW_t$$
$$\alpha(t,T)dt = \sigma(t,T)\int_t^T \sigma(t,s)ds$$

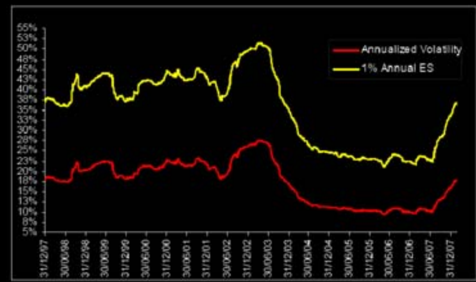
CONSOB

Volatility has a close correspondance with any risk measure, like *Value-at-Risk* (*VaR*) and …



CONSOB

… *Expected Shortfall* (*ES*)



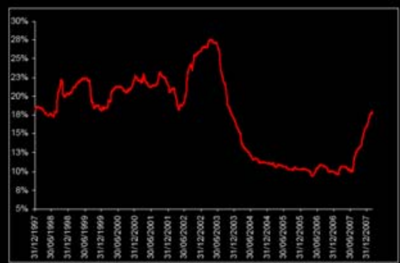
CONSOB

Syllabus

- Volatility
  - Importance and Relationship with other Risk Measures
  - Random Variable and Stochastic Process

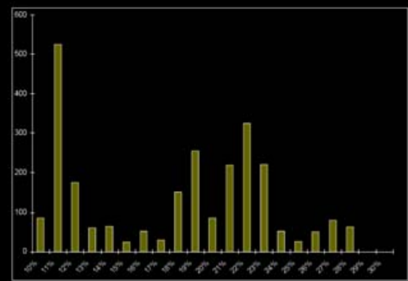
CONSOB

Plot of the Time Series of the Annualized Volatility



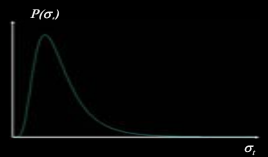
CONSOB

Probability Distribution of the Annualized Volatility



CONSOB

Volatility is
a ***Random Variable***



The Time Series of σ_t is
a ***Stochastic Process***

Syllabus

- **The GARCH Diffusive Approach**
  - Intuition
  - The Convergence Theorem on R[2]
    - The Statement
    - The Conditions
  - The Diffusion Limit of the M-GARCH(1,1)
    - The Statement
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  - The Predictive Interval for the Volatility
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    - Determination of the Interval
  - Other GARCH Models

Need for Volatility Forecasts based on Stochastic Volatility Models

↓

TIME SERIES ANALYSIS OF VOLATILITY

MODELLING THE TIME SERIES OF
VOLATILITY THROUGH
THE DIFFUSION LIMIT OF GARCH PROCESSES

↓

from: STOCHASTIC DIFFERENCE EQUATIONS

to: STOCHASTIC DIFFERENTIAL EQUATIONS

via: SHRINKING of the TIME INTERVALS

Syllabus

- The GARCH Diffusive Approach
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The sequence $\{X_t^h\}$, whose measurable space is $(\mathbb{R}^2, \mathbb{B}(\mathbb{R}^2))$, converges weakly for h ↓0 to the process $\{X_t\}$ which has a unique distribution and is characterized by the following stochastic differential equation:

$$dX_t = b(x,t)dt + \sigma(x,t)dW_{2,t}$$

where $W_{2,t}$ is a two-dimensional standard Brownian motion, if the conditions 1-4, presented below, are satisfied.

The process $\{X_t\}$ has a distribution independent on the choice of $\sigma(x,t)$ and it takes finite values over finite time intervals, i.e. $\forall T > 0$:

$$P\left(\sup_{0 \le t \le T} \|X_t\| < \infty\right) = 1$$

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CONDITION 1

If there exists a δ>0 such that:

$$\lim_{h \downarrow 0} \begin{pmatrix} c_{h,\delta}(x_1,t) \\ c_{h,\delta}(x_2,t) \end{pmatrix} = 0$$

Then there exist $a(x,t)$ and $b(x,t)$, continuous measures respectively mapping from $\mathbb{R}^2 \times [0,\infty)$ into the space of the *2x2* semi-definite positive matrices, and from $\mathbb{R}^2 \times [0,\infty)$ into $\mathbb{R}^2$, such that:

$$\lim_{h \downarrow 0} \begin{pmatrix} b_h(x_1,t) \\ b_h(x_2,t) \end{pmatrix} = \begin{pmatrix} b(x_1,t) \\ b(x_2,t) \end{pmatrix}$$

$$\lim_{h \downarrow 0} \begin{pmatrix} a_h(x_1,t) & a_h((x_1,x_2),t) \\ a_h((x_2,x_1),t) & a_h(x_2,t) \end{pmatrix} = \begin{pmatrix} a(x_1,t) & 0 \\ 0 & a(x_2,t) \end{pmatrix}$$

CONDITION 2

There exists $\sigma(x,t)$, a continuous mapping from $\mathbb{R}^2 \times [0,\infty)$ into $\mathbb{R}^2$, such that, $\forall x_1 \in \mathbb{R}^1, \forall x_2 \in \mathbb{R}^1$, it holds:

$$\begin{pmatrix} \sigma(x_1,t) & 0 \\ 0 & \sigma(x_2,t) \end{pmatrix} = \begin{pmatrix} \sqrt{a(x_1,t)} & 0 \\ 0 & \sqrt{a(x_2,t)} \end{pmatrix}$$

CONDITION 3

For h↓0, X_0^h converges in distribution to a random variable X_0 with probability measure v_0 on $(\mathbb{R}^2, \mathbb{B}(\mathbb{R}^2))$

CONDITION 4

v_0, $a(x,t)$ and $b(x,t)$ uniquely specify the distribution of the process $\{X_t\}$ characterized by an initial distribution v_0, a conditional second moment $a(x,t)$ and a conditional first moment $b(x,t)$

Syllabus

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Given the equation of the conditional variance* in the M-GARCH(1,1):

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2 = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma_k^2 + \beta_1^{(k)}\ln Z_k^2$$

or, equivalently:

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2 = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma_k^2 + 2\beta_1^{(k)}\ln|Z_k|$$

Z_k is i.i.d. *N(0,1)*

its diffusion limit is:

$$d\ln\sigma_t^2 = \left[\beta_0 + 2\beta_1\mathbf{E}(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2\right]dt + 2|\beta_1|\sqrt{Var(\ln|Z_t|)}dW_t^*$$

Z_t is *N(0,1)*

*The focus is on the difference equation for the volatility.
For the convergence of the first equation see Minenna 2003.

Syllabus

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STEP 1:
THE RE-SCALING OF THE PROCESS
The *k* intervals are divided into *1/h* subintervals each one of length *h*

⇓

$$\ln\sigma_{(k+1)h}^2 - \ln\sigma_{kh}^2$$
$$=$$
$$\beta_{0h} + (\beta_{1h} - h)\ln\sigma_{kh}^2 + 2\beta_{1h}\left\{\sqrt{h}\left[\ln|Z_k| - E(\ln|Z_k|)\right] + E(\ln|Z_k|)\right\}$$

STEP 2:
THE CONSTRUCTION OF THE PROCESS $\left\{\ln\sigma_t^{2^h}\right\}$
Definition of the probability measure P_h on the Skorokhod Space D such that:

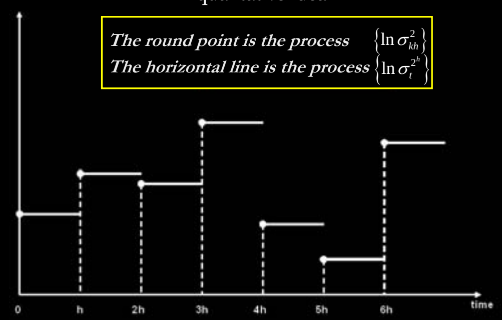
$$P_h(\ln\sigma_0^{2^h} \in \Gamma) = v_0(\Gamma) \qquad \forall\Gamma \in \mathbb{B}(\mathbb{R}^1)$$
$$P_h(\ln\sigma_t^{2^h} = \ln\sigma_{kh}^2, \quad \forall\, kh \le t < (k+1)h) = 1$$
$$P_h(\ln\sigma_{(k+1)h}^2 \in \Gamma|\widehat{\Im}_{kh}) = \Pi_{h,kh}(\ln\sigma_{kh}^2, \Gamma) \text{ a.s. under } P_h, \forall k \ge 0, \forall\Gamma \in \mathbb{B}(\mathbb{R}^1)$$

⇓

$$\ln\sigma_{t+1}^{2^h} - \ln\sigma_t^{2^h}$$
$$=$$
$$\beta_{0h} + (\beta_{1h} - h)\ln\sigma_t^{2^h} + 2\beta_{1h}\left\{\sqrt{h}\left[\ln|Z_t^h| - E(\ln|Z_t^h|)\right] + E(\ln|Z_t^h|)\right\}$$

A qualitative idea

The round point is the process $\{\ln\sigma_{kh}^2\}$
The horizontal line is the process $\{\ln\sigma_t^{2^h}\}$



STEP 3:
CHECK OF CONDITION 1 OF THE CONVERGENCE THEOREM
Finding the values of β_{0h} and β_{1h} which guarantee the convergence of the conditional moments

$$\left.\begin{aligned} \beta_{0h} &:= \beta_0 \cdot h \\ \beta_{1h} &:= \beta_1 \cdot h \end{aligned}\right\} \Longleftrightarrow \begin{cases} \lim_{h\downarrow 0} c_{h,\delta=1}(\widehat{\ln\sigma^2}, t) = 0 \\ \lim_{h\downarrow 0} b_h(\widehat{\ln\sigma^2}, t) = (\beta_0 + 2\beta_1\mathbf{E}(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2) \\ \lim_{h\downarrow 0} a_h(\widehat{\ln\sigma^2}, t) = 4\beta_1^2 Var(\ln|Z_t|) \end{cases}$$

STEP 4:
CHECK OF CONDITIONS 2, 3 and 4 OF THE CONVERGENCE THEOREM

- Condition 2 is verified for every $\sigma>0$, i.e.:

$$\sigma(\widehat{\ln \sigma^2}, t) = 2\,|\beta_1|\,\sqrt{Var(\ln|Z_t|)}$$

- Condition 3 is evidently satisfied by construction of the process $\left\{\ln \sigma_t^{2\wedge}\right\}$
- Consequently, Condition 4 is verified too.

Q.E.D.

Syllabus

KEY POINT

From the Diffusion Limit of the GARCH Process it is possible to establish a *Predictive Interval for σ_t*

Syllabus

$$d\ln\sigma_t^2 = \left[\beta_0 + 2\beta_1 \mathbf{E}(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2\right]dt + 2\,|\beta_1|\,\sqrt{Var(\ln|Z_t|)}\,dW_t^*$$

$$\ln\sigma_t^2 \sim N\left[\left(\ln\sigma_{t-1}^2 + \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)} - \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)};\ \sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)}\right]$$

Syllabus

The relationship between the Stochastic Difference Equation and the Stochastic Differential Equation

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2 = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma_k^2 + 2\beta_1^{(k)}\ln|Z_k|$$

$\Updownarrow$

$$d\ln\sigma_t^2 = \left[\beta_0 + 2\beta_1 \mathbf{E}(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2\right]dt + 2\,|\beta_1|\,\sqrt{Var(\ln|Z_t|)}\,dW_t^*$$

Matching of the first two Conditional Moments

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2$$
$$=$$
$$\left(e^{(\beta_1-1)}-1\right)\left(\frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right) + \left(e^{(\beta_1-1)}-1\right)\ln\sigma_k^2 + 2\left(e^{(\beta_1-1)}-1\right)\ln|Z_k|$$

The Maximum Likelihood Method

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2 = \widehat{a} + \widehat{b}\ln\sigma_k^2 + e_k$$

$$\beta_0 = f_1\left(\widehat{a}, \widehat{b}\right)$$

$$\beta_1 = f_2\left(\widehat{a}, \widehat{b}\right)$$

$$2\,|\beta_1|\,\sqrt{Var\left(\ln|Z_t|\right)} = f_3\left(\widehat{a}, \widehat{b}, e_k\right)$$

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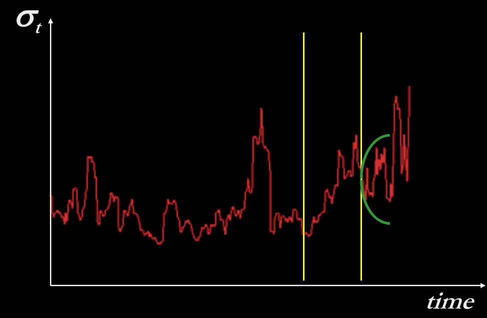
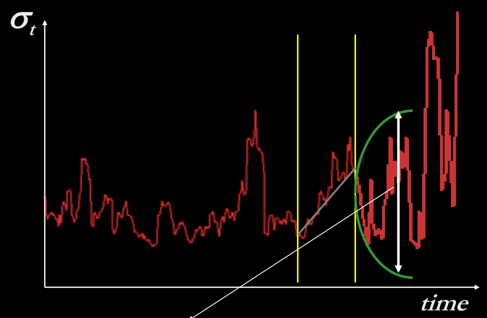
$$P\left(\begin{array}{c} -z_{\frac{\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)}+\left(\ln\sigma_{t-1}^2+\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)} \\ \le \ln\sigma_t^2 \le \\ z_{\frac{\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)}+\left(\ln\sigma_{t-1}^2+\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0+2\beta_1E(\ln|Z_t|)}{(\beta_1-1)} \end{array}\right)=\alpha$$

$$\left[\sigma_{t,\min}^G,\sigma_{t,\max}^G\right]=\left[e^{\frac{-z_{\frac{\alpha}{2}}\sqrt{\frac{(2.2214|\beta_1|)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)}+\left(\ln\sigma_{t-1}^2+\frac{\beta_0-1.2704\beta_1}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0-1.2704\beta_1}{(\beta_1-1)}}{2}};\ e^{\frac{z_{\frac{\alpha}{2}}\sqrt{\frac{(2.2214|\beta_1|)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)}+\left(\ln\sigma_{t-1}^2+\frac{\beta_0-1.2704\beta_1}{(\beta_1-1)}\right)e^{(\beta_1-1)}-\frac{\beta_0-1.2704\beta_1}{(\beta_1-1)}}{2}}\right]$$





Width of the Predictive Interval

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Analogous Procedure

THE DIFFUSION LIMIT OF THE L-GARCH(1,1)

Given the L-GARCH(1,1) model:

$$\sigma_{k+1}^2-\sigma_k^2=\omega+\sigma_k^2(\beta+\varpi Z_k^2-1)$$

Z_k is $N(0,1)$

its diffusion limit is:

$$d\sigma_t^2=[\omega+\vartheta\sigma_t^2]dt+\sqrt{2\varpi}\sigma_t^2dW_t$$

Analogous Procedure

THE DIFFUSION LIMIT OF THE L-GARCH(1,1)

Given the L-GARCH(1,1) model:

$$\sigma_{k+1}^2-\sigma_k^2=\omega+\sigma_k^2(\beta+\varpi Z_k^2-1)$$

Z_k is $N(0,1)$

its diffusion limit is:

$$d\sigma_t^2=[\omega+\vartheta\sigma_t^2]dt+\sqrt{2\varpi}\sigma_t^2dW_t$$

THE DIFFUSION LIMIT OF THE E-GARCH(1,1)

Given the E-GARCH(1,1) model:

$$\ln\sigma_{k+1}^2-\ln\sigma_k^2 = \beta_0^{(k)}+\beta_1^{(k)}\ln\sigma_k^2+\beta_2^{(k)}(|Z_k|+\vartheta Z_k)$$

Z_k is $N(0,1)$

its diffusion limit is:

$$d\ln\sigma_t^2 = \left[\alpha_0+\frac{2}{\sqrt{2\pi}}\left(\alpha_4+\frac{\alpha_5}{2}\right)-\alpha_4-\alpha_5-\alpha_1-1+(\alpha_1-1)\ln\sigma_t^2\right]dt -\frac{\alpha_5}{2}dW_t+\left|\alpha_4+\frac{\alpha_5}{2}\right|\sqrt{\frac{\pi-2}{\pi}}dW_t^*$$

Syllabus

- Application to Flexible Funds Risk Assessment
  - Key Concepts on Flexible Funds
  - Transparency Regulation on the Risk Profile
  - The Perspective of the Asset Manager
  - Quantitative Methodology for Risk Measurement
  - The Solution for the Asset Manager
  - Migration and Prospectus

DEFINITION	Freedom to invest in any market and in any financial instrument and to take leveraged positions
OBJECTIVE	Maximization of the expected return for a given level of risk

Asset Classes selected by the Asset Manager

Use of derivatives and leverage

***Risk* Exposure**

- **Application to Flexible Funds Risk Assessment**
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QUALITATIVE RISK CLASSES
Low
Medium-Low
Medium
Medium-High
High
Very High

Fundamental Assumption

The quantitative risk assessment

is based on

VOLATILITY MEASURES

Variation of the Volatility over time

Exposure to the ***Migration Risk***

Variation of the Volatility over time

Exposure to the ***Migration Risk***

QUALITATIVE RISK CLASSES
Low
Medium-Low
Medium
Medium-High
High
Very High

Assessment and Delimitation of the *Migration Risk*

Regulatory framework consistent with the markets evolution and the activity of the Asset Manager

- **Application to Flexible Funds Risk Assessment**
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Evaluation and Management of the *Migration Risk*

Position inside the Risk Class declared in the Prospectus

Syllabus

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Mapping of the Qualitative Risk Classes into corresponding Volatility Intervals

Step 1: Definition of the Loss Intervals of the Fund

What is the Loss in a Financial Investment?

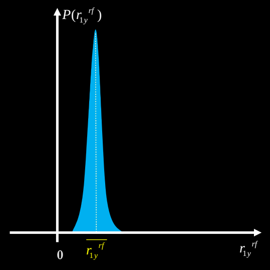
RISK-NEUTRALITY PRINCIPLE

$$\text{LOSS} \in (-100\%, \overline{r^{rf}}\,]$$

where: $\overline{r^{rf}}$ = average of the Probability Distribution of the risk-free rate

Step 1: Definition of the Loss Intervals of the Fund

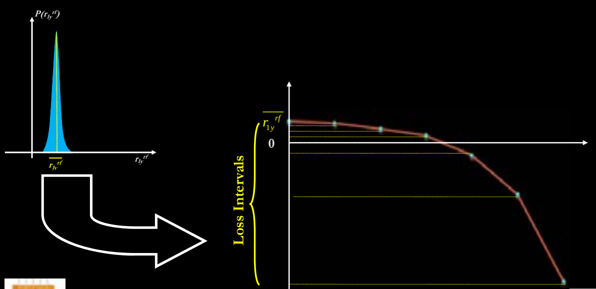
After having selected the Probability Distribution of the 1-year risk-free rate …



$\overline{r_{1y}^{rf}}$ = average of the Probability Distribution of the 1-year risk-free rate

Step 1: Definition of the Loss Intervals of the Fund

… to each Qualitative Risk Class is associated the corresponding annual Loss Interval (multiple of r_{1y}^{rf} according to an exponential function) …



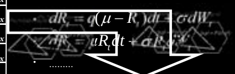
Step 1: Definition of the Loss Intervals of the Fund

… obtaining six initial loss intervals:

Risk Classes	Loss Intervals L_{min}	L_{max}
low	${}_0L_{1,min}$	${}_0L_{1,max}$
medium-low	${}_0L_{2,min}$	${}_0L_{2,max}$
medium	${}_0L_{3,min}$	${}_0L_{3,max}$
medium-high	${}_0L_{4,min}$	${}_0L_{4,max}$
high	${}_0L_{5,min}$	${}_0L_{5,max}$
very high	${}_0L_{6,min}$	${}_0L_{6,max}$

Step 2: Mapping of the Loss Intervals of the Fund to the corresponding Volatility Intervals of the Fund

Risk Classes	Loss Intervals L_{min}	L_{max}
low	${}_0L_{1,min}$	${}_0L_{1,max}$
medium-low	${}_0L_{2,min}$	${}_0L_{2,max}$
medium	${}_0L_{3,min}$	${}_0L_{3,max}$
medium-high	${}_0L_{4,min}$	${}_0L_{4,max}$
high	${}_0L_{5,min}$	${}_0L_{5,max}$
very high	${}_0L_{6,min}$	${}_0L_{6,max}$



Risk Classes	Volatility Intervals σ_{min}	σ_{max}
low	${}_0\sigma_{1,min}$	${}_0\sigma_{1,max}$
medium-low	${}_0\sigma_{2,min}$	${}_0\sigma_{2,max}$
medium	${}_0\sigma_{3,min}$	${}_0\sigma_{3,max}$
medium -high	${}_0\sigma_{4,min}$	${}_0\sigma_{4,max}$
high	${}_0\sigma_{5,min}$	${}_0\sigma_{5,max}$
very high	${}_0\sigma_{6,min}$	${}_0\sigma_{6,max}$

*The subscript *0* preceding the volatility indicates that this is the initial interval, i.e. before the calibration

Step 3: Calibration of the Intervals

REQUIREMENTS

- *Ability to express in a robust and significant way the risk level "typical" of the corresponding Qualitative Class*
- *Stability over time also when the yield curve changes significantly*

Step 3: Calibration of the Intervals

REQUIREMENTS

- *Ability to express in a robust and significant way the risk level "typical" of the corresponding Qualitative Class*
- *Stability over time also when the yield curve changes significantly*

TOOLS

- *GARCH Diffusive Models*
- *Stochastic Non-Linear Programming*

Step 3: Calibration of the Intervals

REQUIREMENTS

- *Ability to express in a robust and significant way the risk level "typical" of the corresponding Qualitative Class*
- *Stability over time also when the yield curve changes significantly*

TOOLS

- *GARCH Diffusive Models*
- *Stochastic Non-Linear Programming*

Fine-tuning Intervention on the Volatility Intervals

Step 3: Calibration of the Intervals

3.0 *Selection of an initial Volatility Interval*

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
low	${}_0\sigma_{1,min}$	${}_0\sigma_{1,max}$
medium-low	${}_0\sigma_{2,min}$	${}_0\sigma_{2,max}$
medium	${}_0\sigma_{3,min}$	${}_0\sigma_{3,max}$
medium-high	${}_0\sigma_{4,min}$	${}_0\sigma_{4,max}$
high	${}_0\sigma_{5,min}$	${}_0\sigma_{5,max}$
very high	${}_0\sigma_{6,min}$	${}_0\sigma_{6,max}$

$\Rightarrow [{}_0\sigma_{4,min} \quad {}_0\sigma_{4,max}]$

Step 3: Calibration of the Intervals

3.1 *Simulation of the Fund pattern*

NAV **Stochastic Differential Equation**

Step 3: Calibration of the Intervals

3.1 *Simulation of the Fund pattern*

NAV S.D.E. ⇒ What Parameters?

Step 3: Calibration of the Intervals

3.1 *Simulation of the Fund pattern*

NAV S.D.E. ⇒ What Parameters?

The Drift

Step 3: Calibration of the Intervals

3.1 *Simulation of the Fund pattern*

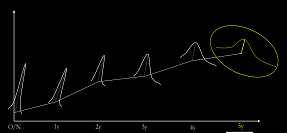
Risk-Neutrality Principle

⇓

Drift $= r^{rf}$

Step 3: Calibration of the Intervals

3.1 *Simulation of the Fund pattern*



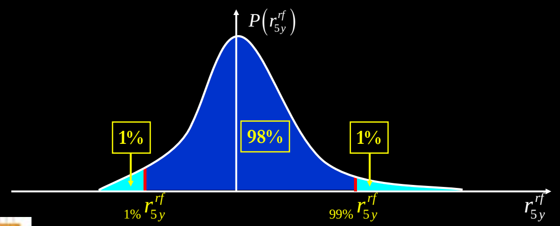
Robustness of the Volatility Intervals

⇓

Drift $= r^{rf}_{5y}$

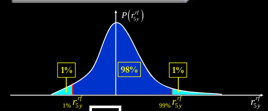
Step 3: Calibration of the Intervals

3.1 *Simulation of the Fund pattern*

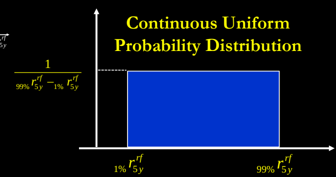


Step 3: Calibration of the Intervals

3.1 *Simulation of the Fund pattern*



Continuous Uniform Probability Distribution



Step 3: Calibration of the Intervals

3.1 Simulation of the Fund pattern

NAV S.D.E. ⇒ What Parameters?

The Diffusion

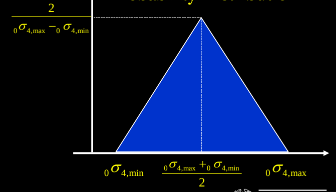
Step 3: Calibration of the Intervals

3.1 Simulation of the Fund pattern

Initial Volatility Interval: $[{}_0\sigma_{4,min}\ \ {}_0\sigma_{4,max}]$

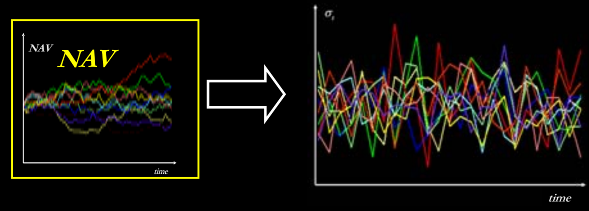
Representativeness of the Volatility Intervals

Symmetric Triangular Probability Distribution



Step 3: Calibration of the Intervals

3.2 Determination of the Time Series of the Annual Volatility



Step 3: Calibration of the Intervals

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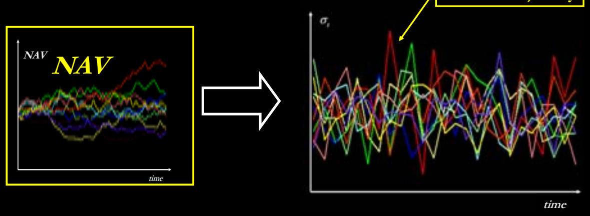
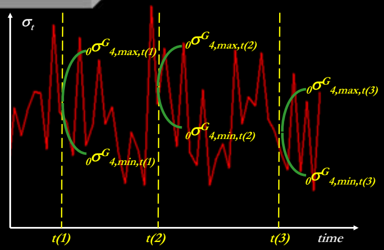


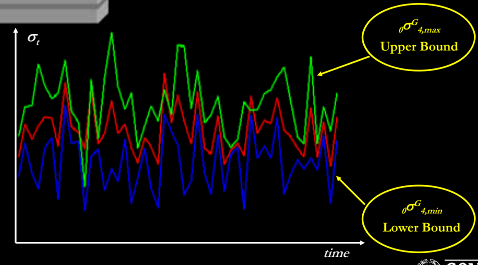
Step 3: Calibration of the Intervals

3.3 Volatility Forecast Band through GARCH Diffusive Models



Step 3: Calibration of the Intervals

3.3 Volatility Forecast Band through GARCH Diffusive Models



Step 3: Calibration of the Intervals

3.4 Validation of the initial Volatility Interval

$[{}_0\sigma_{4,min}\ \ {}_0\sigma_{4,max}]$

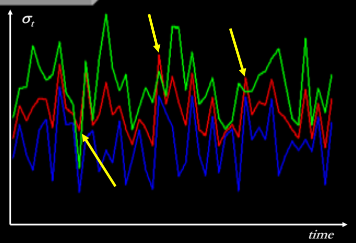
VS

$[{}_0\sigma^G_{4,min}\ \ {}_0\sigma^G_{4,max}]$

Step 3: Calibration of the Intervals

3.4 Validation of the initial Volatility Interval

Calculus of the number of observations outside the Band



Step 3: Calibration of the Intervals

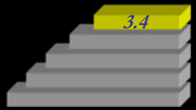
3.4 Validation of the initial Volatility Interval

Calculus of the number of observations outside the Band

Trajectory	n. obs. $\in [{}_0\sigma_{4,min}\ \ {}_0\sigma_{4,max}]$	n. obs. $< {}_0\sigma_{4,min}$	n. obs. $> {}_0\sigma_{4,max}$
1			
2			
…			
n			
	Tot $\in [{}_0\sigma_{4,min}\ \ {}_0\sigma_{4,max}]$	Tot. $< {}_0\sigma_{4,min}$	Tot. $> {}_0\sigma_{4,max}$

Hp.: n. of observations of σ_t = 250 for each trajectory

Step 3: Calibration of the Intervals



Validation of the initial Volatility Interval

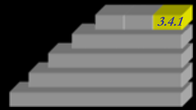
Number of observations outside the Band in 1 year

$$\Delta = \frac{[\text{Tot.} > {}_0\sigma_{4,max}] + [\text{Tot.} < {}_0\sigma_{4,min}]}{n*250}$$

$$\Delta_{up} = \frac{[\text{Tot.} > {}_0\sigma_{4,max}]}{n*250} \qquad \Delta_{down} = \frac{[\text{Tot.} < {}_0\sigma_{4,min}]}{n*250}$$

CONSOB

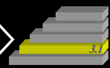
Step 3: Calibration of the Intervals



Update of the initial Volatility Interval

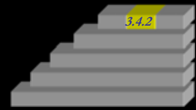
Iteration of the Procedure

$\Delta > 5\%$ and $\Delta_{up} > \Delta_{down}$ → $[{}_0\sigma_{4,min} \quad {}_1\sigma_{4,max}]$ →



CONSOB

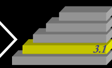
Step 3: Calibration of the Intervals



Update of the initial Volatility Interval

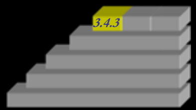
Iteration of the Procedure

$\Delta > 5\%$ and $\Delta_{up} < \Delta_{down}$ → $[{}_1\sigma_{4,min} \quad {}_0\sigma_{4,max}]$ →



CONSOB

Step 3: Calibration of the Intervals



End of the Procedure

$\Delta \leq 5\%$ → $[{}_0\sigma_{4,min} \quad {}_0\sigma_{4,max}] = [\sigma_{4,min} \quad \sigma_{4,max}]$

CONSOB

Step 3: Calibration of the Intervals

OUTPUT

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
low	*0.01%*	*0.49%*
medium-low	*0.50%*	*1.59%*
medium	*1.60%*	*3.99%*
medium-high	*4.00%*	*9.99%*
high	*10.00%*	*24.99%*
very high	*25.00%*	*above 25.00%*

CONSOB

Syllabus

- **Application to Flexible Funds Risk Assessment**
  - Key Concepts on Flexible Funds
  - Transparency Regulation on the Risk Profile
  - The Perspective of the Asset Manager
  - Quantitative Methodology for Risk Measurement
  - The Solution for the Asset Manager
  - Migration and Prospectus

CONSOB

Mapping of the QualitativeRisk Classes to corresponding Volatility Intervals

Risk Classes	Volatility Intervals	
	σ_{min}	σ_{max}
low	*0.01%*	*0.49%*
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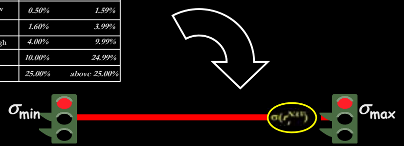

CONSOB

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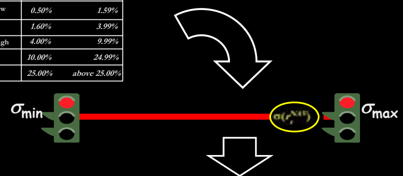

SAFE ASSETS

CONSOB

Mapping of the QualitativeRisk Classes to corresponding Volatility Intervals

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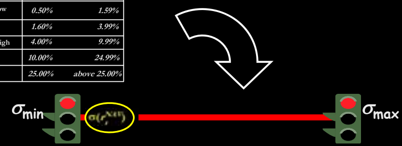

CONSOB

Mapping of the QualitativeRisk Classes to corresponding Volatility Intervals

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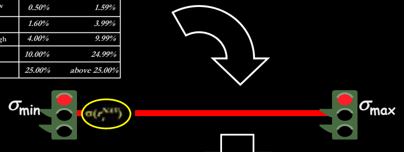


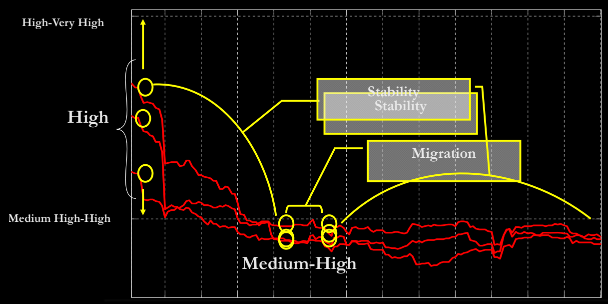
RISKY ASSETS

Syllabus

- **Application to Flexible Funds Risk Assessment**
  - Key Concepts on Flexible Funds
  - Transparency Regulation on the Risk Profile
  - The Perspective of the Asset Manager
  - Quantitative Methodology for Risk Measurement
  - The Solution for the Asset Manager
  - Migration and Prospectus

When does the Migration occur?

The Migration occurs when the fund remains for a significant period outside the qualitative class declared in the Prospectus …

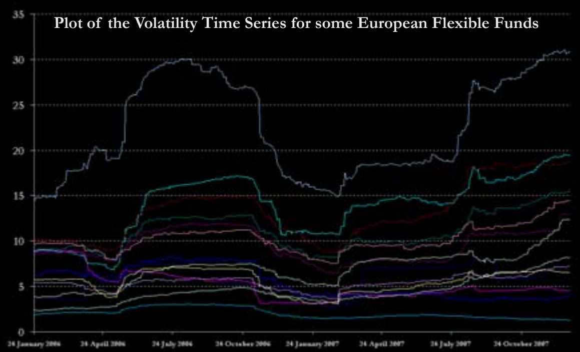


Syllabus

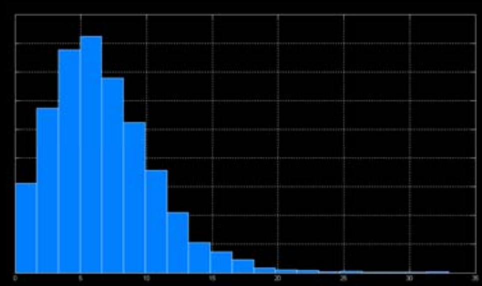
- **Empirical Evidence on the European Industry**
  - Preliminary Informations
  - The Evolution of the Risk-Profile over time

Country	Total (A)	Selected (B)	Representativity (B/A)
Austria	17	13	76.5%
France	92	53	57.6%
Germany	63	45	71.4%
Ireland	2	1	50.0%
Italy	58	52	89.7%
Luxembourg	252	153	60.7%
Spain	224	130	58.0%
UK	8	7	87.5%
Total	716	454	63.4%

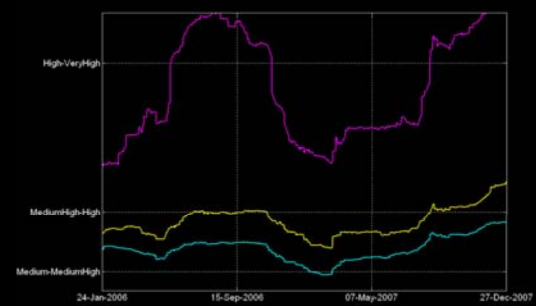
Plot of the Volatility Time Series for some European Flexible Funds



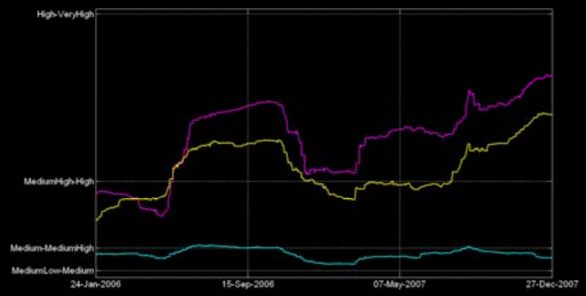
Histogram of the Volatility Time Series of the Flexible Funds selected



LUXEMBOURG

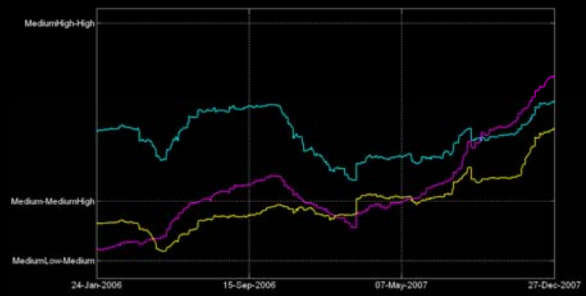


GERMANY



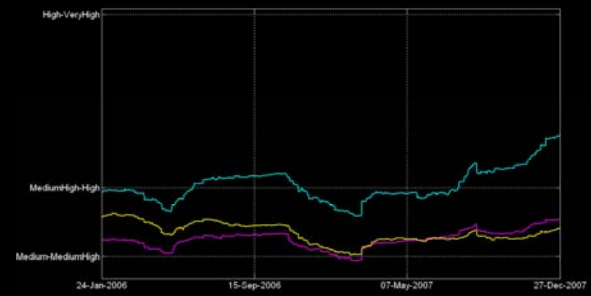
CONSOB

ITALY



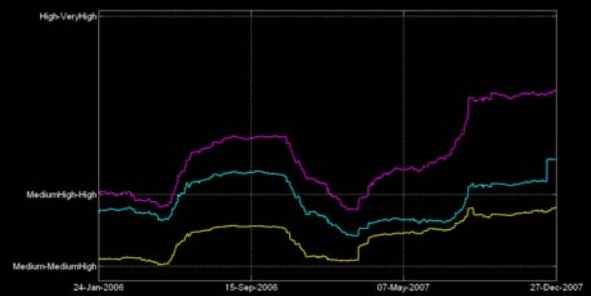
CONSOB

SPAIN



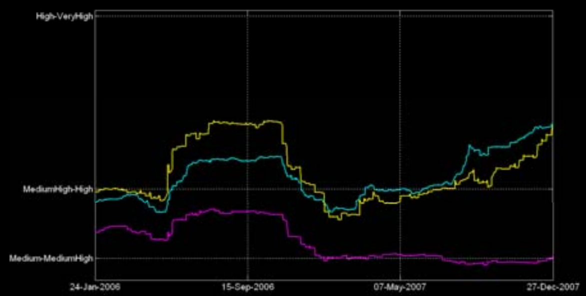
CONSOB

FRANCE



CONSOB

UNITED KINGDOM



CONSOB

Initial Distribution of the 454 Funds between the 6 risk classes (abs. values)

	Initial Risk Class as from 1st January 2006					
Country	1	2	3	4	5	Total
Austria	0	0	4	8	1	13
France	0	2	9	37	5	53
Germany	0	2	10	26	7	45
Ireland	0	1	0	0	0	1
Italy	1	11	11	28	1	52
Luxembourg	1	6	30	100	16	153
Spain	0	23	33	62	12	130
UK	0	0	0	5	2	7
Total	2	45	97	266	44	454

CONSOB

Initial Distribution of the 454 Funds between the 6 risk classes (perc. values)

	Initial Risk Class as from 1st January 2006					
Country	1	2	3	4	5	Total
Austria	0.0%	0.0%	30.8%	61.5%	7.7%	100%
France	0.0%	3.8%	17.0%	69.8%	9.4%	100%
Germany	0.0%	4.4%	22.2%	57.8%	15.6%	100%
Ireland	0.0%	100.0%	0.0%	0.0%	0.0%	100%
Italy	1.9%	21.2%	21.2%	53.8%	1.9%	100%
Luxembourg	0.7%	3.9%	19.6%	65.4%	10.5%	100%
Spain	0.0%	17.7%	25.4%	47.7%	9.2%	100%
UK	0.0%	0.0%	0.0%	71.4%	28.6%	100%
Total	0.4%	9.9%	21.4%	58.6%	9.7%	100%

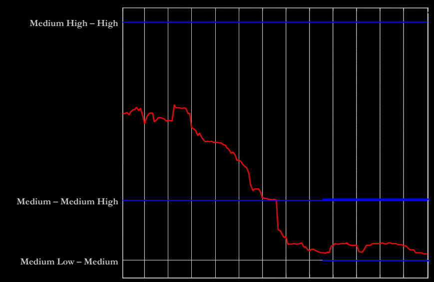
CONSOB

Syllabus

- Empirical Evidence on the Italian industry
  - Preliminary Informations
  - The Evolution of the Risk-Profile over time

CONSOB

MIGRATION



Risk Class as from the Prospectus
Risk effectively taken

CONSOB

Number of Migrations occurred between different risk classes over the period 01/01/2006 – 12/31/2007 (abs. values)

	Number of Migrations over the period January 2006 - December 2007						
Country	*0*	*1*	*2*	*3*	*4*	*5*	*Total*
Austria	*2*	*6*	*2*	*3*	*0*	*0*	13
France	*20*	*13*	*8*	*11*	*1*	*0*	53
Germany	*18*	*6*	*13*	*8*	*0*	*0*	45
Ireland	*0*	*1*	*0*	*0*	*0*	*0*	1
Italy	*15*	*12*	*17*	*8*	*0*	*0*	52
Luxembourg	*63*	*28*	*34*	*23*	*4*	*1*	153
Spain	*44*	*30*	*31*	*21*	*4*	*0*	130
UK	*1*	*3*	*1*	*2*	*0*	*0*	7
Total	163	99	106	76	9	1	454

Analysis of the Migrations *per* Country over the period 01/01/2006 – 12/31/2007 (perc. values)

Country	*0*	*1*	*2*	*3*	*4*	*5*	*Total*
Austria	*15.4%*	46.2%	*15.4%*	*23.1%*	*0.0%*	*0.0%*	*100%*
France	37.7%	*24.5%*	*15.1%*	*20.8%*	*1.9%*	*0.0%*	*100%*
Germany	40.0%	*13.3%*	*28.9%*	*17.8%*	*0.0%*	*0.0%*	*100%*
Ireland	*0.0%*	100.0%	*0.0%*	*0.0%*	*0.0%*	*0.0%*	*100%*
Italy	*28.8%*	*23.1%*	32.7%	*15.4%*	*0.0%*	*0.0%*	*100%*
Luxembourg	41.2%	*18.3%*	*22.2%*	*15.0%*	*2.6%*	*0.7%*	*100%*
Spain	33.8%	*23.1%*	*23.8%*	*16.2%*	*3.1%*	*0.0%*	*100%*
UK	*14.3%*	42.9%	*14.3%*	*28.6%*	*0.0%*	*0.0%*	*100%*

Frequency of the Migrations over the period 01/01/2006 – 12/31/2007 (perc. values)

Country	*0*	*1*	*2*	*3*	*4*	*5*
Austria	*1.2%*	*6.1%*	*1.9%*	*3.9%*	*0.0%*	*0.0%*
France	*12.3%*	*13.1%*	*7.5%*	*14.5%*	*11.1%*	*0.0%*
Germany	*11.0%*	*6.1%*	*12.3%*	*10.5%*	*0.0%*	*0.0%*
Ireland	*0.0%*	*1.0%*	*0.0%*	*0.0%*	*0.0%*	*0.0%*
Italy	*9.2%*	*12.1%*	*16.0%*	*10.5%*	*0.0%*	*0.0%*
Luxembourg	38.7%	*28.3%*	32.1%	30.3%	44.4%	*100.0%*
Spain	*27.0%*	30.3%	*29.2%*	*27.6%*	44.4%	*0.0%*
UK	*0.6%*	*3.0%*	*0.9%*	*2.6%*	*0.0%*	*0.0%*
Total	*100%*	*100%*	*100%*	*100%*	*100%*	*100%*

Conclusions

- ✓ GARCH Diffusive Approach to make robust and reliable Volatility Forecast *(adaptiveness, no echoes effects)*
- ✓ Financial Application to the Transparency regulation of Flexible Mutual Funds
  - mapping of qualitative risk classes to calibrated, increasing and non overlapping intervals of the annualized volatility of NAV returns
  - usefulness of this quantitative methodology to monitor the exposure to the migration risk and to promptly capture the occurrence of the migrations which requires a timely update of the Prospectus.

- ✓ Empirical Evidence:
  the phenomenon of the migration interests more the funds which belong to the riskiest classes
- ✓ Closing Recommendations:
  exploring other fields of application of the described methodology, especially to move faster towards a really levelled playing field

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