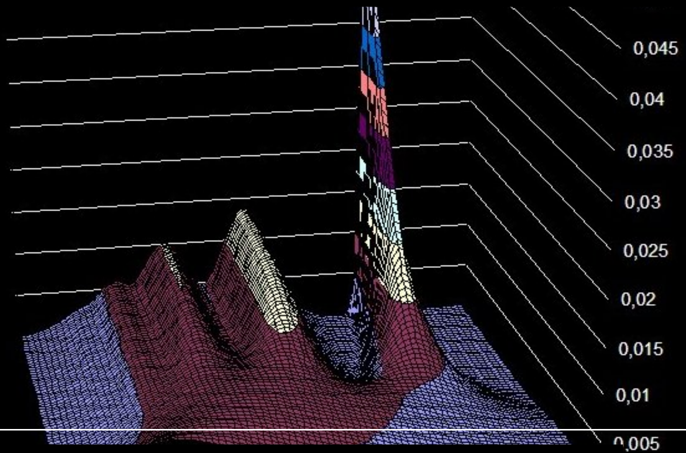
The deep structure of numerical instability inherent to Fourier transform methods in option pricing:

A Gauss-Lobatto FFT approach

Theory and Implementation



Marcello Minenna - Paolo Verzella

Trieste – XXX Congresso AMASES

Syllabus of the presentation

- Review of FFT approach in Option Pricing
- Gauss-Lobatto FFT approach
  - Gauss Lobatto theory
  - Gauss Lobatto option pricing
- Pricing Performance and Calibration

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Review of FFT approach in option pricing

the single integration formula

Carr – Madan (1999)

$$C_0(\ln K) = \frac{e^{-\alpha \ln K}}{\pi}\int_0^{\infty} \Re\left[e^{-iv\ln K}\psi_0(v)\right]dv$$

where:

$$\psi_0(v) = \frac{e^{-rT}\phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}$$

$$\phi_T[q(\ln S_T)](\xi) = \int_{-\infty}^{\infty} e^{i\xi \ln S_T} q(\ln S_T)\, d\ln S_T$$

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Main Features

Only one integral to be computed (doubles speed in FT-Q methods)

Problems of Accuracy reduced of an order of 1/2

Arbitrary choice of a dampening parameter

FFT Implementation

Quadrature Algorithm

a — N-C

b — ?

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Newton – Cotes Schemes

Proposition *The integral of a generic function $f: \mathbb{R} \to \mathbb{R}$ can be approximated in the form:*

$$\int_{\alpha}^{\beta} f(x)\,dx \approx \frac{n}{c} h \sum_{j=0}^{n} a_j^{(i)} \cdot f(x_j)$$

where:

- d *is the grade of the approximating polynomial;*
- $n = i \cdot d$ *– is the number of subintervals of* $i = 1, 2, 3, \ldots N$ *–* $[\alpha, \beta]$;
- $c = \sum_{j=0}^{n} a_j^{(i)}$
- $h = \frac{(\beta - \alpha)}{n}$ *is the dimension if each subinterval;*
- $x_j = \alpha + jh$ *per* $i = 0, 1, \ldots, d$;

Newton – Cotes Schemes

They use a fixed, equally spaced, discretization grid for the characteristic formula

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Higher order of integration = Better Accuracy

Newton – Cotes Schemes

Unfortunately, the characteristic formula is OFTEN an oscillatory function with abrupt changes

So, OFTEN, Newton – Cotes schemes fail in giving accurate and stable prices

Newton – Cotes Schemes

Higher order of integration (greater than 8th) = Numerical Instability

If the characteristic function goes to infinity, the NC schemes EXPLODE

The Pricing via Newton-Cotes Algorithms

Trapezoid Rule

$$C_0\left(\ln K\right)_u \approx \frac{e^{-\alpha\left[\ln S_t - b + \lambda(u-1)\right]}}{\pi}\eta \cdot \cdot \Re\left(\sum_{j=1}^{N} e^{-i\eta\lambda(j-1)(u-1)} e^{-i\eta(j-1)\left[\ln S_t - b\right]}\psi_0\left((j-1)\eta\right)\right)$$

The Pricing via Newton-Cotes Algorithms

Simpson Rule

$$C_0\left[\ln K\right]_u \approx \frac{e^{-\alpha\left[\ln S_t - b + \lambda(u-1)\right]}}{\pi}\frac{\eta}{3} \cdot \cdot \Re\left\{\sum_{j=1}^{N} e^{-i\eta\lambda(j-1)(u-1)} e^{-i(j-1)\eta\left[\ln S_t - b\right]}\psi_0\left((j-1)\eta\right) \cdot \cdot \left(3 + (-1)^j - \delta_{j-1}\right)\right\}$$

NC Rules

Under the following parameters specification (recombinant parameters)

$$\left[\ln K\right]_u = \ln S_t - b + \lambda\left(u-1\right)$$

$$\lambda\eta = \frac{2\pi}{N}$$

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$$[\ln K]_u = \ln S_t - b + \lambda (u-1)$$

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The Call Prices are computed via Cooley-Tukey Algorithm

FFT Implementation

Quadrature Algorithm

a — N-C

b — Gauss

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Gauss Schemes

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The discretization points are choosen in order to fit perfectly a polynomial function

The schemes depend from the choice of an optimization criterion

- **Example: Gauss-Lobatto Quadrature Formula**

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(-1) + w_N f(1) + \sum_{i=2}^{N-1} w_i f(x_i)$$

$$w_i = \frac{2}{N(N-1)\left[P_{N-1}(x_i)\right]^2}$$

LIMITED
to the interval (-1,1)

$$w_1 = w_N = \frac{2}{N(N-1)}$$

where $P_{N-1}(x)$

is a Legendre Polynomial of order ***N-1***

$$y_1(x) = \gamma_0 + \sum_{m=1}^{\infty} \gamma_{2m} x^{2m} \qquad per\ \varphi = 2m$$

$$y_2(x) = \gamma_1 x + \sum_{m=1}^{\infty} \gamma_{2m+1} x^{2m+1} \qquad per\ \varphi = 2m+1$$

The Gautschi - Gander extension (2000)

ENHANCE

The Gauss Lobatto formula

They develop a GL recursive adaptive algorithm for a generic interval

The Gautschi - Gander extension (2000)

$$\int_\alpha^\beta f(x)\,dx \approx h\left\{w_1 f(\alpha) + w_N f(\beta) + \sum_{i=2}^{N-1} w_i \left[f(m + x_i h)\right]\right\}$$

$$w_i = \frac{2}{N(N-1)\left[P_{N-1}(x_i)\right]^2}$$

$$w_1 = w_N = \frac{2}{N(N-1)}$$

$$h = \frac{1}{2}(\beta - \alpha)$$

$$m = \frac{1}{2}(\alpha + \beta)$$

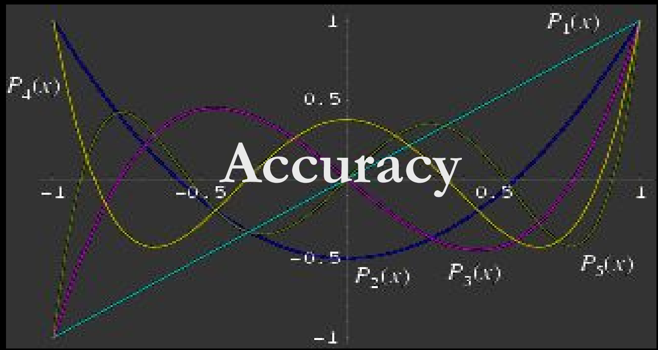
The Extended GL scheme is both

Accurate **Stable**

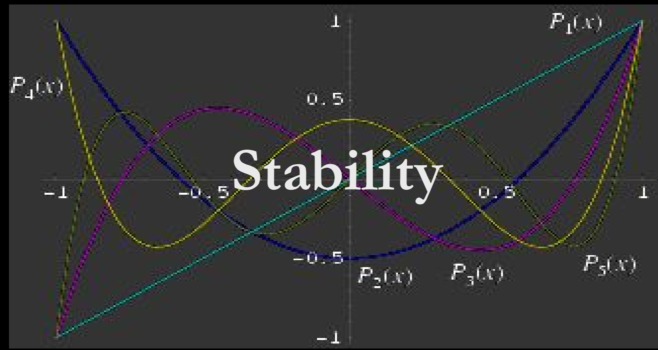
Let's see why

Accuracy

The Fundamental Theorem of Gaussian Quadrature states that the optimal abscissas of the N-point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function.



The Roots of Legendre Polynomials are optimal discretization points



Legendre Polynomials are oscillating functions

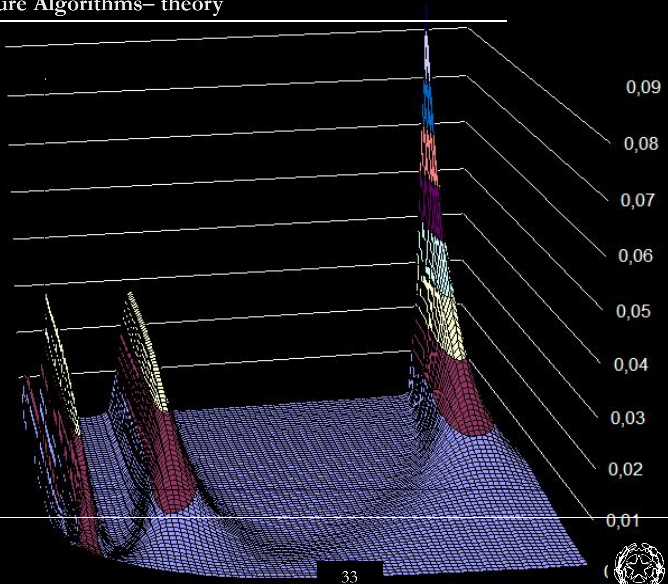
Legendre Polynomials are oscillating functions

Increasing the order of N is useful to fit the oscillatory decay of the characteristic functions

Legendre Polynomials are oscillating functions

Even if great, N remains finite, so the GL schemes cannot EXPLODE when the characteristic function goes to infinity

Quadrature Algorithms– theory



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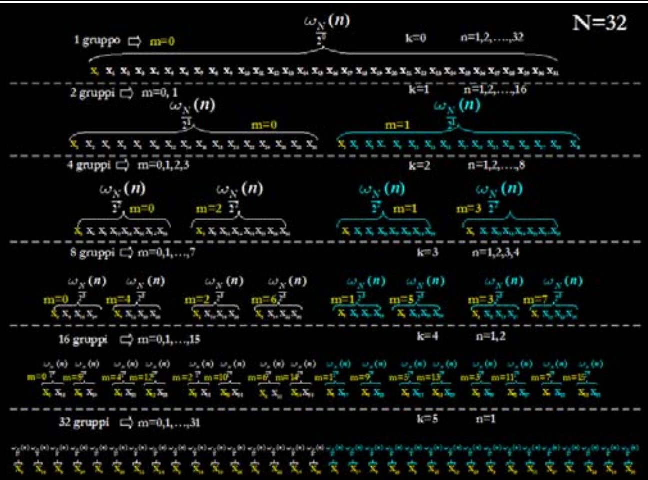
The Pricing via Gauss-Lobatto Algorithm

$$C_0\left(\ln K\right) \simeq \frac{\epsilon^{-\alpha \ln K}}{\pi} a \left\{ \frac{1}{N(N-1)} \left[\Re\left[e^{-iv_0 \ln K} \psi_0\left(v_0\right)\right] + \Re\left[e^{-iv_a \ln K} \psi_0\left(v_a\right)\right]\right] + \right.$$
$$\left. + \sum_{i=2}^{N-1} \frac{1}{N(N-1)[P_{N-1}(x_i)]^2} \Re\left[e^{-i\left(\frac{1}{2}a(1+v_i)\right)\ln K} \psi_0\left(\frac{1}{2}a\left(1+v_i\right)\right)\right] \right\}$$

The Pricing via Gauss-Lobatto Algorithm

It requires a proper readjustment of Cooley-Tukey Algorithm to take care of the variable grid size

Fast Fourier Transform Method – The Pricing via Quadrature Algorithms



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Pricing performance and calibration

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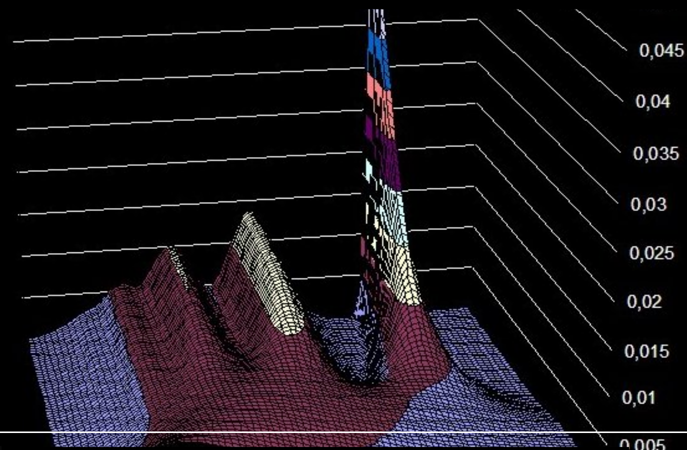

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