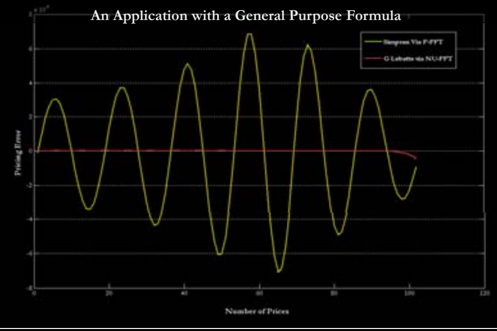
Fractional vs Non Uniform Discrete Fourier Transform

An Application with a General Purpose Formula



Marcello Minenna - Paolo Verzella

Syllabus of the presentation

- Review of Option Pricing via DFT
  - The Lewis Standard Machine
  - DFT Convergence to FT
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids
- Fast Option Pricing
  - Fractional FFT
  - Non Uniform FFT
    - Gaussian Gridding: a matter of interpolation
  - Fractional vs Non Uniform FFT: Empirical Analsysis
  - Conclusions

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The Lewis Standard Machine

$$\tilde{w}(z) = \int_{-\infty}^{+\infty} e^{-izx} w(x)dx$$

is the PayOff functional's Transform

$$\phi_T(z) = E^Q\left[e^{iz\ln S_t}\right]$$

under risk-neutral measure

A linear direct mapping from Fourier Spectral Space

FT

LEWIS REPRESENTATION

The Lewis Standard Machine

$$\tilde{w}(z) = \int_{-\infty}^{+\infty} e^{-izx} w(x)dx$$

is the PayOff functional's Transform

$$\phi_T(z) = E^Q\left[e^{iz\ln S_t}\right]$$

under risk-neutral measure

A linear direct mapping from Fourier Spectral Space

FT

$$V_t = \frac{e^{-r(T-t)}}{2\pi} \int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K} \phi_T(-z) \int_{-\infty}^{+\infty} e^{-izx} w(x)dxdz$$

The Lewis Standard Machine

Knowing $\tilde{w}(z)$

implies reducing the problem to the calculation of a single integral

The Lewis Standard Machine

$z = \xi + i\alpha$

Financial Claim	$w(x)$	$\tilde{w}(x)$
Call Option	$\max[S_T - K, 0]$	$-\frac{K^{iz+1}}{z^2 - iz},\ \alpha > 1$
Put Option	$\max[K - S_T, 0]$	$-\frac{K^{iz+1}}{z^2 - iz},\ \alpha < 0$
Covered Call	$\min[S_T, K]$	$\frac{K^{iz+1}}{z^2 - iz},\ 0 < \alpha < 1$
Money Market	1	$2\pi\delta(k), \alpha \in \mathbb{R}$
Self Quanto Call	$\max[S_T - K, 0] \cdot S_T$	$\frac{K^{2+2iz}}{(zi+1)^{S_T}(zi+2)^{S_T}},\ \alpha < -2$
Power Call	$\max[S_T - K, 0]^d$	$\frac{K^{d(1+iz)}\Gamma(iz)\Gamma(d+1)}{\Gamma(iz+d+1)}, \alpha < -d$

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DFT Convergence to FT

Given the General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}x_j(m-1)} f(x_j) \quad where\ m = 1, 2, \ldots M$$

Given the General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad \text{where } m = 1,2,\ldots M$$

$$M \neq N$$

The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N\to\infty} \sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j, X)$$

$$t_m = \frac{2\pi}{X}(m-1)$$

C_0 via FT

Convergence theorems

$\overrightarrow{C_0}$ via DFT

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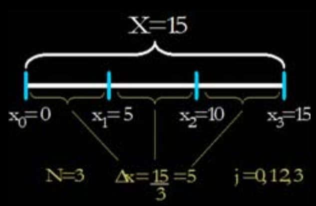
Condition 1

Uniform Discretization Grid



Condition 2

N=M

DFT specialized

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(n-1)} f(x_j) \quad \text{where } n=1,2,\ldots,N$$

Condition 1 Condition 2

DFT Simplified Formula

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i2\pi kj\gamma} f(x_j) \quad \text{where} \quad n = 1\ldots N$$

Nyquist – Shannon Limit (N-S)

$$\mathcal{F}[f(x)](t_n) = \lim_{N\to\infty} \frac{X}{N}\omega(n)$$

$\{t_n\}_{n=1..\frac{N}{2}}$ *for N even*

$\{t_n\}_{n=1..\frac{N+1}{2}}$ *for N odd*

$$C_t = \frac{Ke^{-r(T-t)}}{2\pi} \int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K} \frac{\phi_T(-z)}{z^2-iz} dz$$

Uniform Discretization Grids for ϕ_T

1. $\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]$
2. $\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$

1.

$$C_t = \frac{Ke^{-r(T-t)}}{2\pi}\int_{ia-\infty}^{ia+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_t - b + \lambda(u-1)]}}{b} \cdot \Re(\omega(u))$$

2.

$$C_t = \frac{Ke^{-r(T-t)}}{2\pi}\int_{ia-\infty}^{ia+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_t - b + \lambda(u-1)]}}{3b} \cdot \Re(\omega(u))$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Trapezoid/Simpson Quadrature Rule

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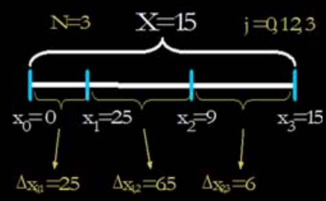
Condition 1

Non Uniform Discretization Grid



Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre

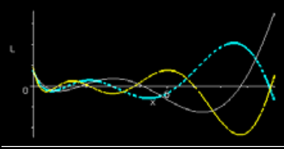
Gander Gautschi

Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre



Zeros of Laguerre Poynomials

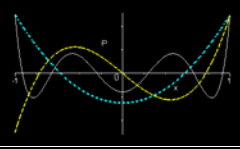
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Lobatto

Zeros of Legendre Poynomials



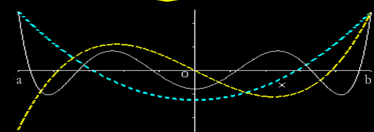
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gander Gautschi

Zeros of rescaled Legendre Poynomials



Condition 2

N≠M

General DFT

$$\omega(m)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \text{ where } m=1,2,\dots,2M$$

The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m)=\lim_{N\to\infty}\sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j,X)$$

$$t_m=\frac{2\pi}{X}(m-1)$$

$$C_t=\frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

Gaussian Grids for ϕ_T

1. $\phi_T(\xi_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a^*}-\ln S_t\right)\right]\xi_{j-1}}\Psi_0[\xi_{j-1}]\cdot\frac{1}{L_{N+1}(\xi_{j-1})L'_N(\xi_{j-1})}$
2. $\phi_T\left(\frac{1}{2}a(1+\xi_{j-1})\right)=e^{\left[-i\left(\frac{1}{2}a(1+\xi_{j-1})\right)\left(\ln S_t-\frac{M\pi}{a^*}\right)\right]}\Psi_0\left[\frac{1}{2}a(1+\xi_{j-1})\right]\cdot\frac{1}{\left[P_{N-1}(\xi_{j-1})\right]^2}$

1.

$$C_t=\frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T(\xi_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a^*}-\ln S_t\right)\right]\xi_{j-1}}\Psi_0[\xi_{j-1}]\cdot\frac{1}{L_{N+1}(\xi_{j-1})L'_N(\xi_{j-1})}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx -\Re\left[\frac{e^{-a\left(\ln S_t-\frac{M\pi}{a^*}+\frac{2\pi}{a^*}(u-1)\right)}}{\pi}\frac{1}{N+1}\cdot\omega^*(u)\right]$$

2.

$$C_t=\frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T\left(\frac{1}{2}a(1+\xi_{j-1})\right)=e^{\left[-i\left(\frac{1}{2}a(1+\xi_{j-1})\right)\left(\ln S_t-\frac{M\pi}{a^*}\right)\right]}\Psi_0\left[\frac{1}{2}a(1+\xi_{j-1})\right]\cdot\frac{1}{\left[P_{N-1}(\xi_{j-1})\right]^2}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx \Re\left[\frac{e^{-a\left(\ln S_t-\frac{M\pi}{a^*}+\frac{2\pi}{a^*}(u-1)\right)}}{\pi}\frac{1}{N(N-1)}\cdot\omega^*\left(\tfrac{1}{2}a(1+v_{j-1})\right)\right]$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Gauss Laguerre/Gander Gautschi Quadrature Rule

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$\overrightarrow{C_t}$ via DFT

allows

Fast Fourier Trasform Algorithms

$\overrightarrow{C_t}$ via DFT

Newton-Cotes

Fractional FFT

$\overrightarrow{C_t}$ via DFT

Gauss

NonUniform FFT

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Bayley-Swarztrauber F-DFT Characterization

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i2\pi kj\gamma} f(x_j) \quad where \quad n = 1 \ldots N$$

Bayley-Swarztrauber F-DFT Characterization

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i2\pi kj\gamma} f(x_j) \quad where \quad n = 1 \ldots N$$

with γ that can be any complex number

If $\gamma = \frac{1}{N}$

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-1)} f(x_j) \quad where\ n = 1, 2, \ldots N$$

The standard DFT definition

Choosing two indipendent uniform grids

$x_j = jg\left(\frac{a}{N}\right) \quad for \quad j = 1 \ldots N$	Spectral Grid
$[\ln K]_u^+ = \ln S_t - b + \lambda_u \quad for \quad u = 1, \ldots, N$	Log-Strike Grid

Choosing two indipendent uniform grids

Implies choosing a specific value of γ

$$\gamma = \frac{\lambda g\left(\frac{a}{N}\right)}{2\pi}$$

Fast Fractional Reconstruction

Step 1

Calculate sequences of $2N$ points

Fast Fractional Reconstruction

Step 1

$$y = \left\{ \left(f\left((j-1)g\left(\frac{a}{N}\right)\right) e^{-i\pi j^2 \gamma} \right)_{j=0}^{N-1}, (0)_{j=0}^{N-1} \right\}$$

$$z = \left\{ \left(e^{i\pi j^2 \gamma} \right)_{j=0}^{N-1}, \left(e^{i\pi (N-j)^2 \gamma} \right)_{j=0}^{N-1} \right\}$$

Fast Fractional Reconstruction

Step 2

Calculate

$$w = \psi_0\left((j-1)g\left(\tfrac{a}{N}\right)\right) \odot \left(e^{i\pi(N-j)^2\gamma}\right)_{j=0}^{N-1}$$

Fast Fractional Reconstruction

Step 3

Calculate via standard FFT

$$\overline{w} = FFT(w),\ \overline{z} = FFT(z)$$

Fast Fractional Reconstruction

Step 4

Calculate

$$q = \overline{w} \odot \overline{z}$$

Fast Fractional Reconstruction

Step 4

Calculate

$$\omega(n) = IFFT(q) \oslash \left(e^{i\pi j^2\gamma}\right)_{j=0}^{N-1}$$

Fast Fractional Reconstruction

The total computational cost drops

from

$$O(N^2)$$

to

$$O(6N\log_2 N)$$

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Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$

Gaussian Gridding

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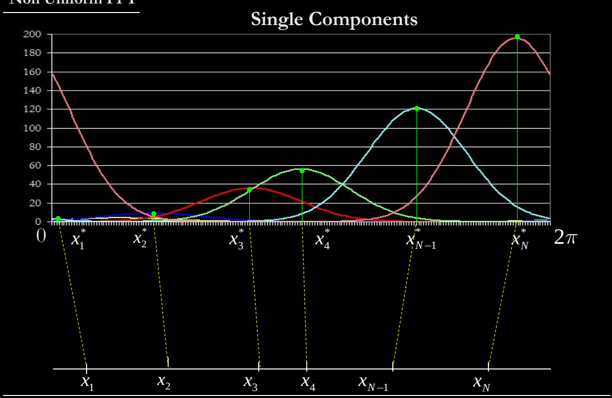

Gaussian Gridding

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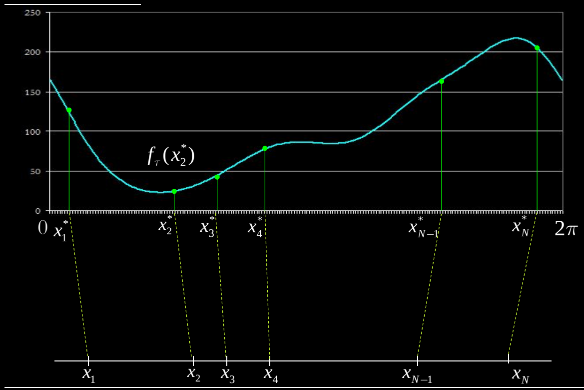


Gaussian Gridding

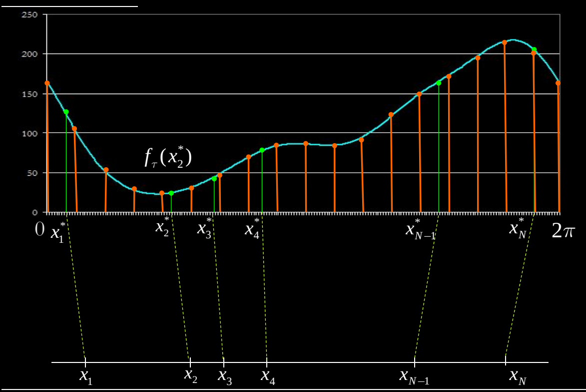
Step 2

Discretization on an uniform oversampled grid of $f_\tau(x)$

$$\widetilde{f}_\tau(y_m) = \sum_{j=0}^{N-1} f(x_j)\, \widetilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi)$$

where

$$\widetilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi) = \begin{cases} \sum_{k=-\infty}^{\infty} e^{-\frac{\left(2\pi\frac{j}{N} - 2\pi\frac{m}{M_\tau} - 2\pi k\right)^2}{4\tau}} \\ \text{oppure} \\ \sum_{k=-\infty}^{\infty} e^{-\frac{\left(x_j^* - 2\pi\frac{m}{M_\tau} - 2\pi k\right)^2}{4\tau}} \end{cases}$$



Gaussian Gridding

Step 3

Computation of the Fourier Coefficient of $f_\tau(x)$ discretised

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \sum_{m=0}^{M_\tau - 1} \widetilde{f}_\tau\left(m \frac{2\pi}{M_\tau}\right) e^{-im\frac{2\pi}{M_\tau}(n-1)}$$

Gaussian Gridding

Step 4

NU-DFT representation of the Fourier Coefficient $F_\tau(n)$

$$\widetilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2\tau} F_\tau(n)$$

Gaussian Gridding

Step 5

DFT representation of the Fourier Coefficient $F_\tau(n)$

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, ..., \frac{M_\tau}{2}$

Gaussian Gridding

Step 6

NU-DFT derivation as a function of DFT

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

$$\widetilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2\tau} F_\tau(n)$$

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2\tau^*} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, ..., \frac{M_\tau}{2}$

Gaussian Gridding

Step 7

NU-FFT computation

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2\tau^*} \frac{1}{M_\tau} \omega(n)$$

FFT

Gaussian Gridding

Step 7
NU-FFT computation

$$\tilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

NU-FFT ------ FFT

Computational Cost

The major computational cost of the Procedure is the FFT on the oversampled grid

Choosing the oversampling ratio

$M_\tau = 2M$

The total cost of the procedure is $\simeq 2M \log 2M$

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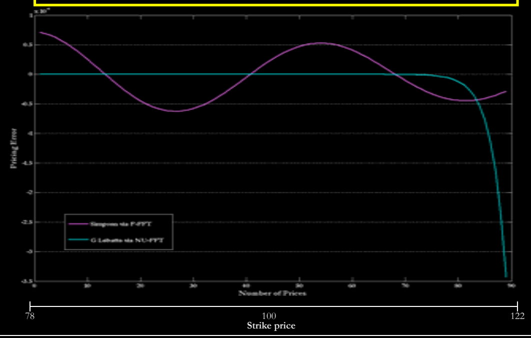
ACCURACY

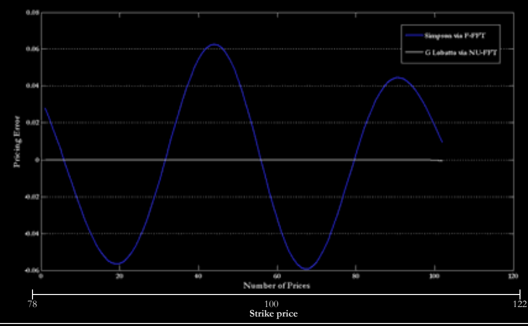
$\sigma = 0.3$

BLACK SCHOLES MODEL



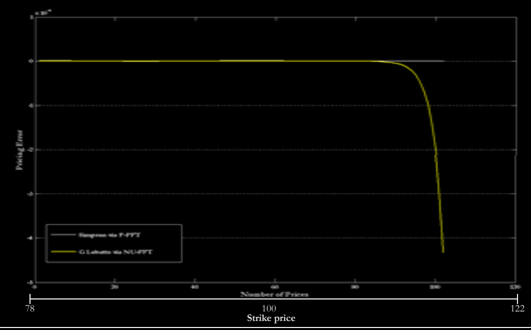
$\sigma = 0.1$

BLACK SCHOLES MODEL

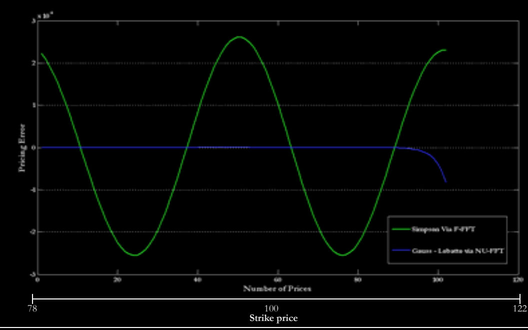


STABILITY

BLACK SCHOLES MODEL

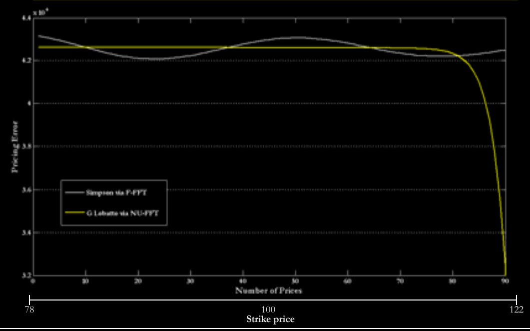


HESTON MODEL



Empirical Analysis

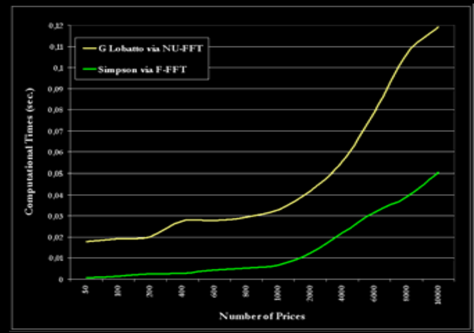
MERTON MODEL



Empirical Analysis

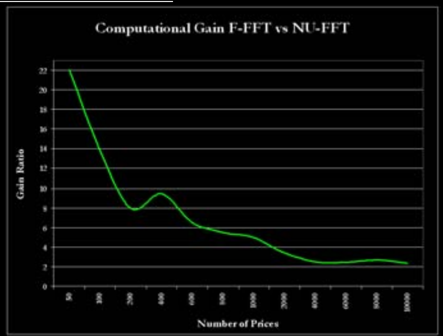
SPEED

Empirical Analysis



Centrino 1600Mhz – 1gb RAM
Mean Value over 1000 runs

The Computational Framework



Centrino 1600Mhz – 1gb RAM
Mean Value over 1000 runs

Empirical Analysis

At very low time scales, the differences are negligible

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Conclusions

Use of Gaussian Grids

F-FFT	NO
NU – FFT	YES

Conclusions

Indifference to Nyquist-Shannon Limit

F-FFT	YES
NU – FFT	YES

Conclusions

Indipendent Price Grids

F-FFT	YES
NU – FFT	YES

FFT's like - Accuracy

F-FFT	YES
NU – FFT	YES

Stability of Pricing

F-FFT	NO
NU – FFT	YES

Speed of Pricing

F-FFT	YES
NU – FFT	YES

	F-FFT	NU – FFT
Gaussian Grids		■
NS Limit	■	■
Indipendent Grids	■	■
Accuracy	■	■
Stability		■
Speed	■	■