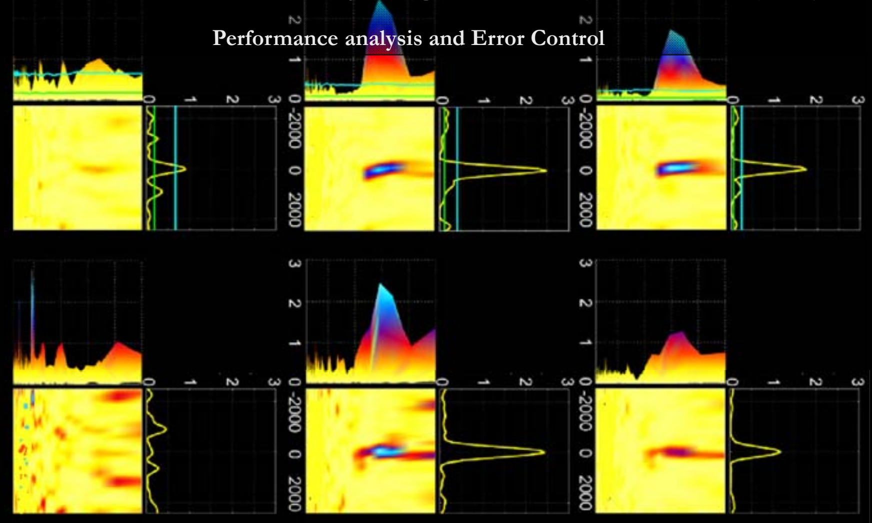
# DFT Methods for Option Pricing
# Fast Extensions on Non Uniform Gaussian Grids

Performance analysis and Error Control



Marcello Minenna - Paolo Verzella





## Syllabus of the presentation

- Review of Option Pricing via DFT
  - FT Pricing formula
  - DFT Convergence to FT
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids
- Fast Option Pricing
  - FFT
  - Non Uniform FFT
    - Gaussian Gridding: a matter of interpolation
    - The Computational Framework: Speed, Stability, Accuracy
- Conclusions



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## FT Pricing Formulas

European Call Price $C_t$

Spot Price $S_t$

$$f_2(\ln S_T, \xi \mid \ln S_0) = \int_{-\infty}^{+\infty} e^{i\xi \ln S_T} q_2(\ln S_T \mid \ln S_0) d \ln S_T$$

under risk-neutral measure

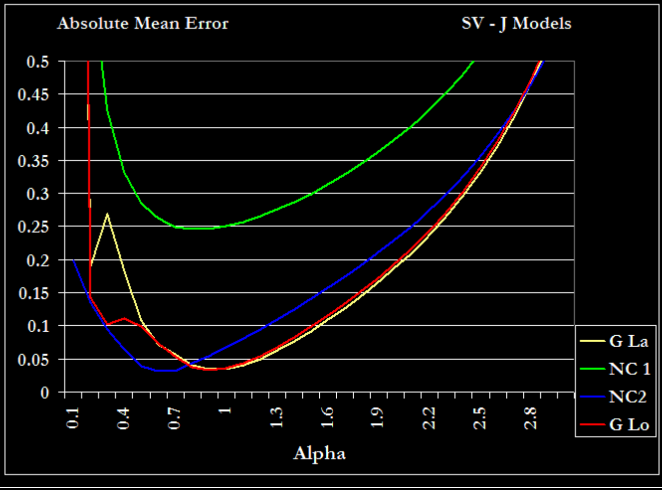
A linear direct mapping from Fourier Space

FT

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re \left( e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \right) d\xi$$

## Accuracy

Absolute Mean Error computed w.r.t. $\alpha$ on an $(\sigma, \tau)$ space

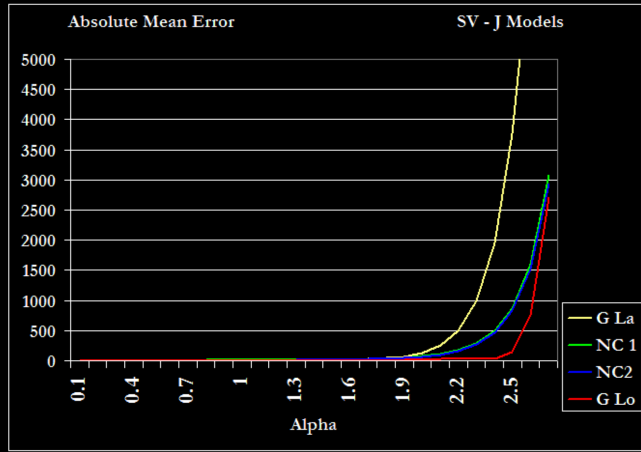







## Stability

Absolute Mean Error computed w.r.t. $\alpha$ on an <u>Extended</u> $(\sigma, \tau)$ space





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## Given the General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad where\ m = 1,2,\ldots M$$

$$M \neq N$$

## The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N \to \infty} \sum_{j=1}^{N} e^{-i \frac{2\pi}{X} x_j (m-1)} f(x_j, X)$$

$$t_m = \frac{2\pi}{X}(m-1)$$





$C_0$ via FT

Convergence theorems

$\overrightarrow{C_0}$ via DFT





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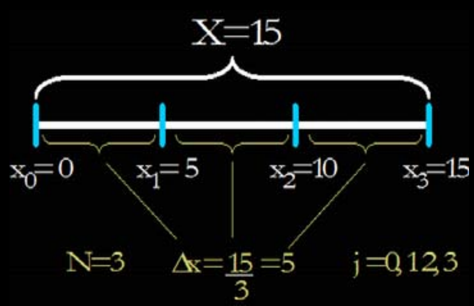
## Condition 1

### Uniform Discretization Grid



## Condition 2

**N=M**

## DFT specialized

$$\omega(n)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(n-1)} f(x_j) \quad \text{where } n=1,2,\ldots,N$$







## Condition 1 Condition 2

## DFT Simplified Formula

$$\omega(n)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-1)} f(x_j) \quad \text{where } n = 1,2,\ldots N$$







## Nyquist – Shannon Limit (N-S)

$$\mathcal{F}[f(x)](t_n)=\lim_{N\to\infty}\frac{X}{N}\omega(n)$$

$$\{t_n\}_{n=1..\frac{N}{2}} \qquad \text{for } N \text{ even}$$

$$\{t_n\}_{n=1..\frac{N+1}{2}} \qquad \text{for } N \text{ odd}$$







$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

## Uniform Discretization Grids for *f*

1. $f(v_{j-1})=e^{-i(j-1)\eta[\ln S_t-b]}\Psi_0[(j-1)\eta]$
2. $f(v_{j-1})=e^{-i(j-1)\eta[\ln S_t-b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$

1.

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i)}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_t - b+\lambda(u-1)]}}{b}\cdot\Re(\omega(u))$$





2.

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i)}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_t - b+\lambda(u-1)]}}{3b}\cdot\Re(\omega(u))$$





Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Trapezoid/Simpson Quadrature Rule



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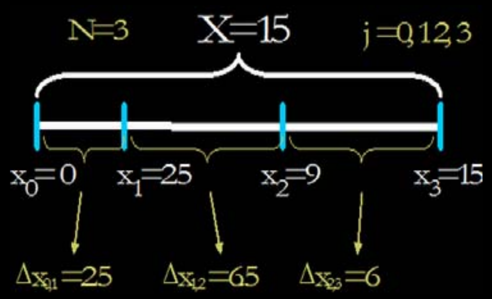
## Condition 1

### Non Uniform Discretization Grid



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## Condition 1

### Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre

Gander Gautschi



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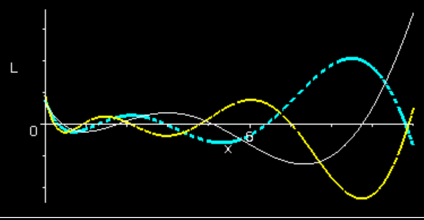
## Condition 1

### Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre




Zeros of
Laguerre Poynomials



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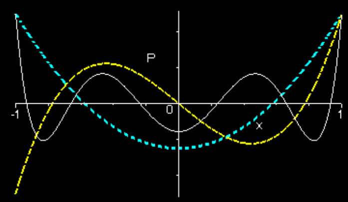
## Condition 1

### Gaussian Grids

Optimal choice of discretization points

Gauss Lobatto

Zeros of
Legendre Poynomials



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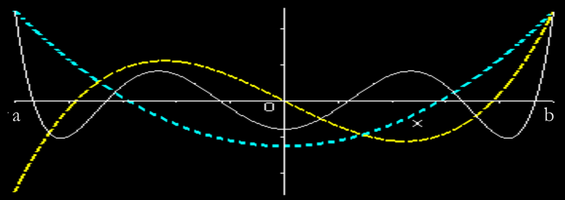
## Condition 1

### Gaussian Grids

### Optimal choice of discretization points

**Gander Gautschi**

**Zeros of rescaled Legendre Poynomials**



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## Condition 2

### N≠M

### General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad \text{where } m=1,2,\ldots,2M$$



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## The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N\to\infty} \sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j, X)$$

$$t_m = \frac{2\pi}{X}(m-1)$$



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$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left( e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi} \right) d\xi$$

### Gaussian Grids for *f*

1. $f(v_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a*}-\ln S_t\right)\right]v_{j-1}} \psi_0(v_{j-1}) \quad \frac{1}{L_{N+1}(v_{j-1})L'_N(v_{j-1})}$

2. $f\left(\frac{1}{2}a(1+v_{j-1})\right) = \left[e^{-i\left(\frac{1}{2}a(1+v_{j-1})\right)\left[\ln S_t-\frac{M\pi}{a}\right]} \psi_0\left(\frac{1}{2}a(1+v_{j-1})\right)\right] \frac{1}{[P_{N-1}(v_{j-1})]^2}$



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1.

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left( e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \right) d\xi$$

$$f(v_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a*} - \ln S_t\right)\right] v_{j-1}} \psi_0(v_{j-1}) \quad \frac{1}{L_{N+1}(v_{j-1}) L'_N(v_{j-1})}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx -\Re\left[ \frac{e^{-\alpha\left(\ln S_t - \frac{M\pi}{a*} + \frac{2\pi}{a*}(u-1)\right)}}{\pi} \frac{1}{N+1} \cdot \omega^*(u) \right]$$



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2.

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left( e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \right) d\xi$$

$$f\left(\frac{1}{2}a(1+v_{j-1})\right) = \left[ e^{-i\left(\frac{1}{2}a(1+v_{j-1})\right)\left[\ln S_t - \frac{M\pi}{a}\right]} \psi_0\left(\frac{1}{2}a(1+v_{j-1})\right) \right] \frac{1}{[P_{N-1}(v_{j-1})]^2}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx \Re\left[ \frac{e^{-\alpha\left(\ln S_t - \frac{M\pi}{a*} + \frac{2\pi}{a*}(u-1)\right)}}{\pi} \frac{1}{N(N-1)} \cdot \omega^*\left(\tfrac{1}{2}a(1+v_{j-1})\right) \right]$$



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**Theorems of Equivalence**

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Gauss Laguerre/Gander Gautschi Quadrature Rule



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## $\overrightarrow{C_t}$ via DFT

allows

Fast Fourier Trasform Algorithms





## $\overrightarrow{C_t}$ via DFT

Newton-Cotes

Uniform FFT





## $\overrightarrow{C_t}$ via DFT

Gauss

NonUniform FFT





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## Cooley-Tukey DFT Characterization

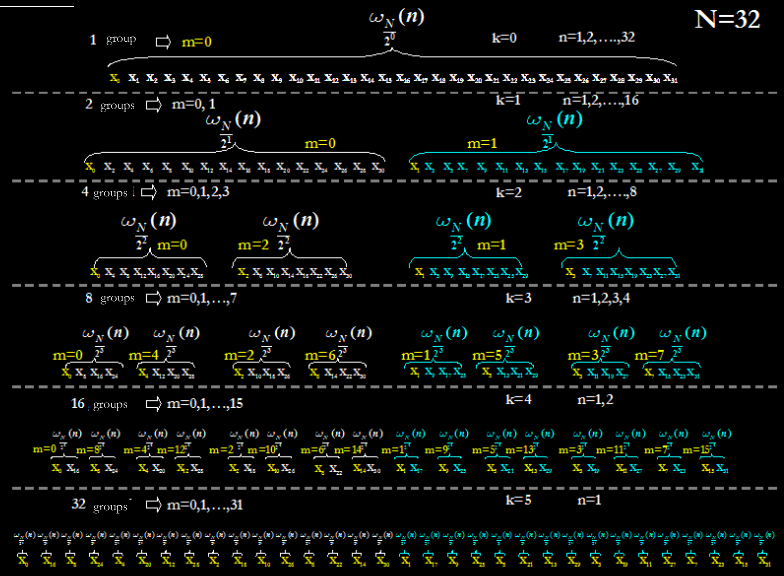
$$\omega_2^m(n) = f(x_m) + W_2^{(n-1)} f(x_{m+N/2}) \quad for\ n = 1,2$$

Iterated Bottom – Up for N stages

It gives the FFT Cooley – Tukey Algorithm












## FFT Cooley – Tukey Algorithm

The DFT computational cost drops

from

$O(N^2)$

to

$O(N \log_2 N)$





## Since the Nyquist – Shannon Limit, the pricing formulas

via FFT

Give accurate prices

**ONLY**

Around the Nyquist Frequency

Approx. 25% of prices can be accepted

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## Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$





## Gaussian Gridding

Step 1

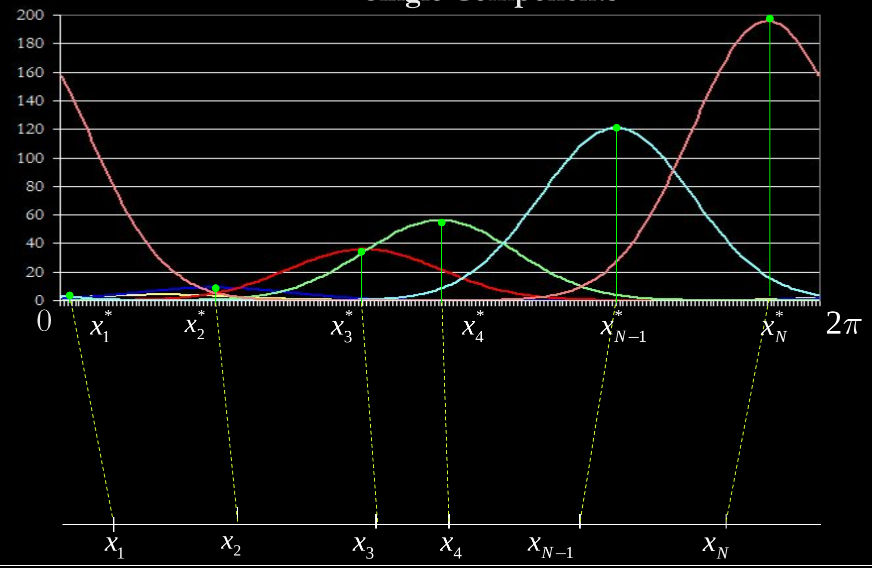
Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \boxed{\sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}}$$





Single Components



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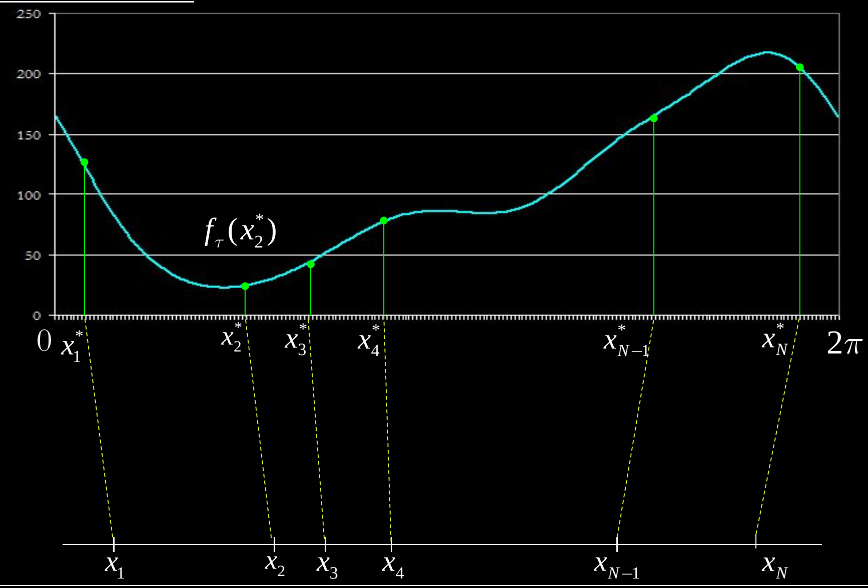







# Gaussian Gridding

Step 2

Discretization on an uniform oversampled grid of $f_\tau(x)$

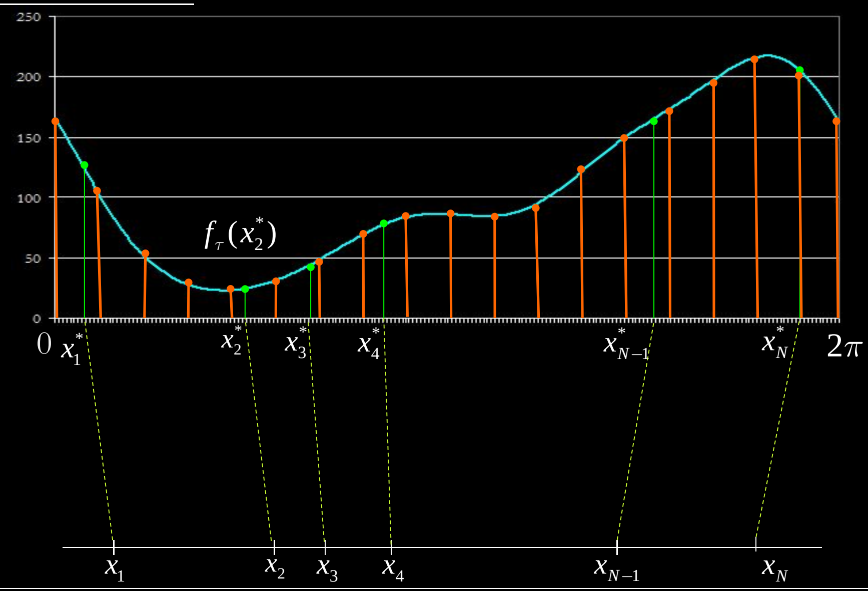
$$\widetilde{f}_\tau(y_m) = \sum_{j=0}^{N-1} f(x_j)\, \widetilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi)$$

where

$$\widetilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi) = \begin{cases} \sum_{k=-\infty}^{\infty} e^{-\frac{\left(2\pi \frac{x_j}{X} - 2\pi \frac{m}{M_r} - 2\pi k\right)^2}{4\tau}} \\ oppure \\ \sum_{k=-\infty}^{\infty} e^{-\frac{\left(x_j^* - 2\pi \frac{m}{M_r} - 2\pi k\right)^2}{4\tau}} \end{cases}$$







# Gaussian Gridding

**Step 3**

**Computation of the Fourier Coefficient of $f_\tau(x)$ discretised**

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \sum_{m=0}^{M_\tau - 1} \widetilde{f}_\tau \left( m \frac{2\pi}{M_\tau} \right) e^{-im\frac{2\pi}{M_\tau}(n-1)}$$







# Gaussian Gridding

**Step 4**

**NU-DFT representation of the Fourier Coefficient $F_\tau(n)$**

$$\widetilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2 \tau} F_\tau(n)$$







# Gaussian Gridding

**Step 5**

**DFT representation of the Fourier Coefficient $F_\tau(n)$**

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, \ldots, \frac{M_\tau}{2}$







# Gaussian Gridding

**Step 6**

**NU-DFT derivation as a function of DFT**

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

$$\widetilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2 \tau} F_\tau(n)$$

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, \ldots, \frac{M_\tau}{2}$

## Gaussian Gridding

Step 7
NU-FFT computation

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

FFT







## Gaussian Gridding

Step 7
NU-FFT computation

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

NU-FFT - - - FFT







## Computational Cost

The major computational cost of the Procedure is the FFT on the oversampled grid

Choosing the oversampling ratio
$M_\tau = 2M$

The total cost of the procedure is $\simeq 2M \log 2M$







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# ACCURACY

G LA
G LO



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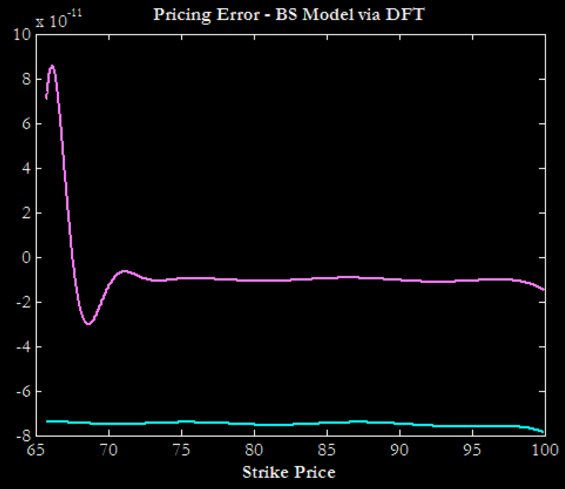


G LA
G LO

2000 Prices computed



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# STABILITY

G LA
G LO



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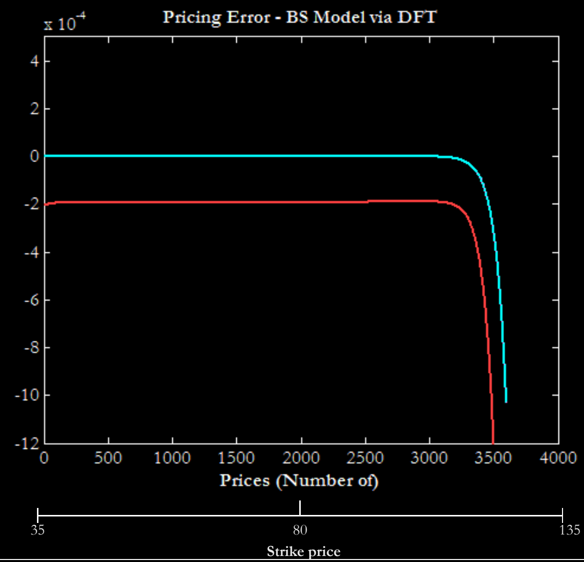


G LA
G LO



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## STABILITY

The error of 90% of prices computed lies in the

$10^{-3}$

**RANGE OF PRECISION**





# SPEED






## SPEED

the NU – FFT is around 2 time slower than FFT





## SPEED

At very low time scales, the differences disappear

## SPEED

At very low time scales, the differences disappear

| | | | |
|---|---|---|---|
| FFT | NC2<br>0.01 sec. | G – LA<br>N/A | G - LO<br>N/A |
| NU – FFT | NC2<br>0.02 sec. | G - LA<br>0.0261 sec. | G - LO<br>0.0301 sec. |

Computation of 4000 prices on a Centrino 1600Mhz – 2gb RAM
Mean Value over 1000 runs





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- NU – FFT allows the use of Gaussian Grids
- NU – FFT is indifferent to Nyquist _Shannon Limit
- NU – FFT is at least as accurate as FFT
- NU – FFT is more stable than FFT
- NU – FFT speed performances are indistinguishable from FFT's ones





**NU – FFT**
is a natural candidate for operational use on trading desks