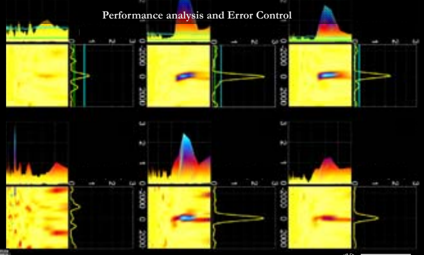
DFT Methods for Option Pricing
Fast Extensions on Non Uniform Gaussian Grids

Performance analysis and Error Control



Marcello Minenna – Paolo Verzella
STRUCTURED PRODUCTS EUROPE – Nov 13, 2007 - London

Syllabus of the presentation

- Review of Option Pricing via DFT
  - FT Pricing formula
  - DFT Convergence to FT
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids
- Fast Option Pricing
  - FFT
  - Non Uniform FFT
    - Gaussian Gridding: a matter of interpolation
    - The Computational Framework: Speed, Stability, Accuracy
- Conclusions

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FT Pricing Formulas

European Call Price C_t

Spot Price S_t

$$f_2(\ln S_T,\xi\,|\,\ln S_0)=\int_{-\infty}^{+\infty} e^{i\xi\ln S_T} q_2(\ln S_T\,|\,\ln S_0)\,d\ln S_T$$

under risk-neutral measure

A linear direct mapping from Fourier Space

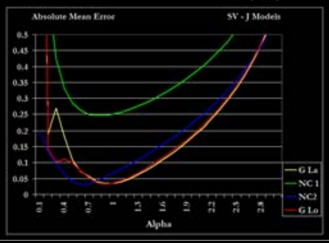
FT

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i)}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

FT Pricing Formulas

Accuracy

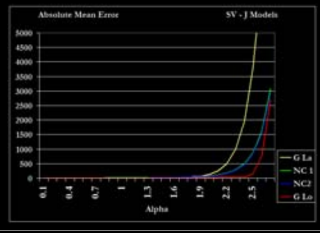
Absolute Mean Error computed w.r.t. α on an (σ,τ) space



FT Pricing Formulas

Stability

Absolute Mean Error computed w.r.t. α on an Extended (σ,τ) space



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DFT Convergence to FT

Given the General DFT

$$\omega(m)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)}f(x_j)\quad where\ m=1,2,\ldots M$$

$M \neq N$

DFT Convergence to FT

The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m)=\lim_{N\to\infty}\sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)}f(x_j,X)$$

$$t_m=\frac{2\pi}{X}(m-1)$$

DFT Convergence to FT

C_0 via FT

Convergence theorems

$\overrightarrow{C_0}$ via DFT

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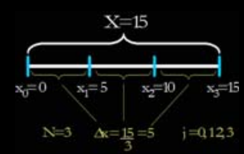
Convergence Theorems for Uniform Grids

Condition 1

Uniform Discretization Grid



Convergence Theorems for Uniform Grids

Condition 2

N=M

DFT specialized

$$\omega(n)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(n-1)}f(x_j)\quad where\ n=1,2,\ldots,N$$

Convergence Theorems for Uniform Grids

Condition 1 Condition 2

DFT Simplified Formula

$$\omega(n)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-1)}f(x_j)\quad where\ n=1,2,\ldots N$$

Convergence Theorems for Uniform Grids

Nyquist – Shannon Limit (N-S)

$$\mathcal{F}[f(x)](t_n)=\lim_{N\to\infty}\frac{X}{N}\omega(n)$$

$\{t_n\}_{n=1\ldots\frac{N}{2}}$ *for N even*

$\{t_n\}_{n=1\ldots\frac{N+1}{2}}$ *for N odd*

Convergence Theorems for Uniform Grids

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i)}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

Uniform Discretization Grids for *f*

1. $f(v_{j-1})=e^{-i(j-1)\eta[\ln S_t-b]}\psi_0((j-1)\eta)$
2. $f(v_{j-1})=e^{-i(j-1)\eta[\ln S_t-b]}\psi_0((j-1)\eta)\cdot\left(3+(-1)^j-\delta_{j-1}\right)$

1.

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(\upsilon_{j-1})=e^{-i(j-1)\eta[\ln S_T-b]}\psi_0((j-1)\eta)$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_T-b+\lambda(u-1)]}}{b}\cdot\Re(\omega(u))$$

2.

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1})=e^{-i(j-1)\eta[\ln S_T-b]}\psi_0((j-1)\eta)\cdot\left(3+(-1)^j-\delta_{j-1}\right)$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_T-b+\lambda(u-1)]}}{3b}\cdot\Re(\omega(u))$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Trapezoid/Simpson Quadrature Rule

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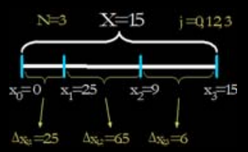
Condition 1

Non Uniform Discretization Grid



Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre

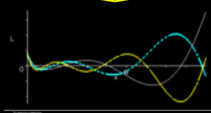
Gander Gautschi

Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre



Zeros of Laguerre Poynomials

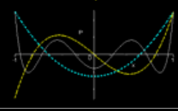
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Lobatto

Zeros of Legendre Poynomials



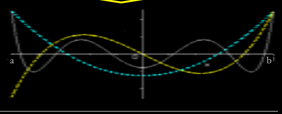
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gander Gautschi

Zeros of rescaled Legendre Poynomials



Condition 2

N≠M

General DFT

$$\omega(m)=\sum_{j=0}^{N-1}e^{-i\frac{2\pi}{X}x_j(m-1)}f(x_j)\quad \text{where } m=1,2,\ldots,2M$$

The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m)=\lim_{N\to\infty}\sum_{j=1}^{N}e^{-i\frac{2\pi}{X}x_j(m-1)}f(x_j,X)$$

$$t_m=\frac{2\pi}{X}(m-1)$$

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

Gaussian Grids for *f*

1. $f(\upsilon_{j-1})=e^{\left[1+i\left(\frac{Mr}{a}-\ln S_t\right)\right]\upsilon_{j-1}}\psi_0(\upsilon_{j-1})\ \frac{1}{L_{N+1}(\upsilon_{j-1})L_N'(\upsilon_{j-1})}$
2. $f\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)=\left[e^{-i\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\left[\ln S_t-\frac{Mr}{a}\right]}\psi_0\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\right]\frac{1}{[P_{N-1}(\upsilon_{j-1})]^2}$

1.

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(\upsilon_{j-1})=e^{\left[1+i\left(\frac{Mr}{a}-\ln S_t\right)\right]\upsilon_{j-1}}\psi_0(\upsilon_{j-1})\ \frac{1}{L_{N+1}(\upsilon_{j-1})L_N'(\upsilon_{j-1})}$$

+

C-Th

$$C_0([\ln K]_u^*)\approx-\Re\left[\frac{e^{-\alpha\left(\ln S_T-\frac{Mr}{a}+\frac{2\pi}{Ma}(u-1)\right)}}{\pi}\frac{1}{N+1}\cdot\omega^*(u)\right]$$

2.

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)=\left[e^{-i\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\left[\ln S_t-\frac{Mr}{a}\right]}\psi_0\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\right]\frac{1}{[P_{N-1}(\upsilon_{j-1})]^2}$$

+

C-Th

$$C_0([\ln K]_u^*)\approx\Re\left[\frac{e^{-\alpha\left(\ln S_T-\frac{Mr}{a}+\frac{2\pi}{Ma}(u-1)\right)}}{\pi}\frac{1}{N(N-1)}\cdot\omega^*\left(\tfrac{1}{2}a(1+\upsilon_{j-1})\right)\right]$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Gauss Laguerre/Gander Gautschi Quadrature Rule

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Fast Option Pricing

$\vec{C}_t$ via DFT

allows

Fast Fourier Trasform Algorithms

Fast Option Pricing

$\vec{C}_t$ via DFT

Newton-Cotes

Uniform FFT

Fast Option Pricing

$\vec{C}_t$ via DFT

Gauss

NonUniform FFT

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Uniform FFT

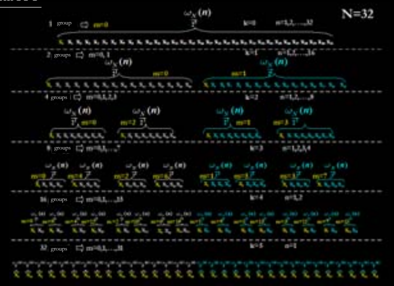
Cooley-Tukey DFT Characterization

$$\omega_2^m(n) = f(x_m) + W_2^{(n-1)} f(x_{m+N/2}) \quad for\ n = 1,2$$

Iterated Bottom – Up for N stages

It gives the FFT Cooley – Tukey Algorithm

Uniform FFT



Uniform FFT

FFT Cooley – Tukey Algorithm

The DFT computational cost drops

from

$O(N^2)$

to

$O(N \log_2 N)$

Uniform FFT

Since the Nyquist – Shannon Limit, the pricing formulas

via FFT

Give accurate prices

ONLY

Around the Nyquist Frequency

Approx. 25% of prices can be accepted

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Non Uniform FFT

Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$

Non Uniform FFT

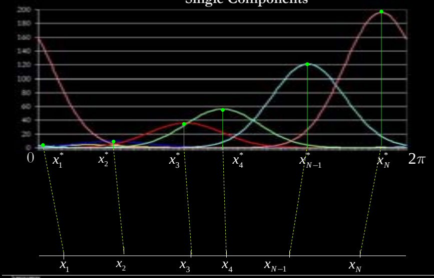
Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \boxed{\sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}}$$

Non Uniform FFT



Non Uniform FFT

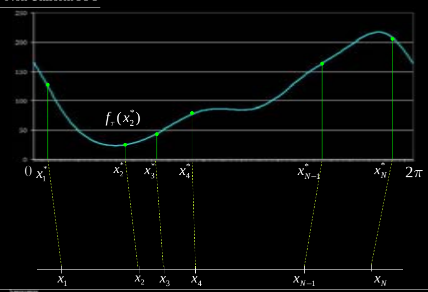
Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$\boxed{f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}}$$

Non Uniform FFT



Non Uniform FFT

Gaussian Gridding

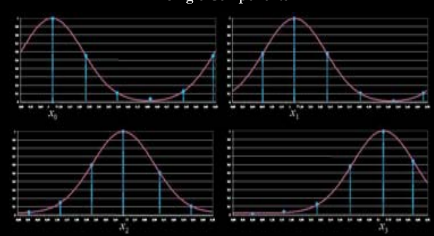
Step 2

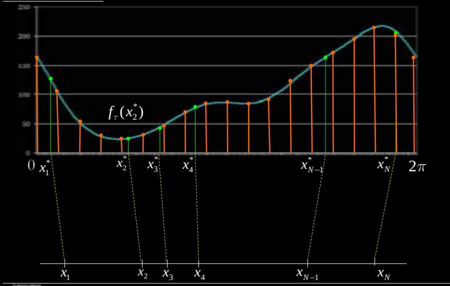
FFT computation on a uniform oversampled grid of the Fourier Coefficient of the convolved characteristic function

$$F_\tau(n) = \frac{1}{2\pi}\int_0^{2\pi} f_\tau(x) e^{-ix(n-1)} dx$$

Non Uniform FFT

Single Components





Gaussian Gridding

Step 3

Elimination of frequencies greater than Nyquist – Shannon Limit

Gaussian Gridding

Step 4

homothetic
rescaling from Gaussian scale

Gaussian Gridding

Step 4

homothetic
rescaling from Gaussian scale

$$\omega(n) = \sqrt{\frac{\pi}{\tau}}\, e^{n^2\tau} F_\tau(n)$$

Computational Cost

The major computational cost of the Procedure is the FFT on the oversampled grid

Choosing the oversampling ratio

$M_r = 2M$

The total cost of the procedure is $\approx 2M \log 2M$

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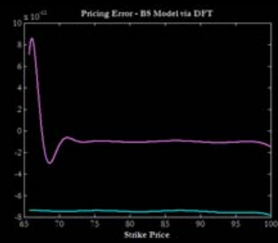
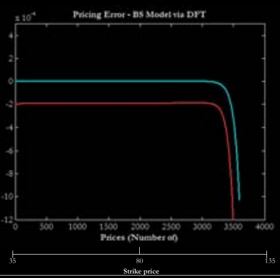
ACCURACY



2000 Prices computed

STABILITY



STABILITY

The error of **90%** of prices computed lies in the

10^{-3}

RANGE OF PRECISION

SPEED

SPEED

the NU – FFT is around
2 time slower than FFT

SPEED

At very low time scales, the differences **disappear**

SPEED

At very low time scales, the differences **disappear**

FFT	NC2	G – LA	G - LO
	0.01 sec.	N/A	N/A
NU – FFT	NC2	G - LA	G - LO
	0.02 sec.	0.0261 sec.	0.0301 sec.

Computation of 4000 prices on a Centrino 1600Mhz – 2gb RAM
Mean Value over 1000 runs

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- NU – FFT allows the use of Gaussian Grids
- NU – FFT is indifferent to Nyquist _Shannon Limit
- NU – FFT is at least as accurate as FFT
- NU – FFT is more stable than FFT
- **NU – FFT speed performances are indistinguishable from FFT's ones**

NU – FFT
is a natural candidate for
operational use on trading desks