# Advanced Solutions in Semianalytical Option Pricing

An Application of Fractional and Non Uniform Discrete Transforms


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Structured products — CONSOB

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## Syllabus of the presentation

- Review of Option Pricing via DFT
  - The Lewis Standard Machine
  - DFT Convergence to FT
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids
- Fast Option Pricing
  - Fractional FFT
  - Non Uniform FFT
    - Gaussian Gridding: a matter of interpolation
  - Fractional vs Non Uniform FFT: Empirical Analysis
  - Conclusions



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## The Lewis Standard Machine

$$\widetilde{w}(z) = \int_{-\infty}^{+\infty} e^{-izx} w(x)dx$$

is the PayOff functional's Transform

Derivative Price $V_t$



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## The Lewis Standard Machine

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$$\phi_T(z) = E^Q\left[e^{iz\ln S_T}\right]$$

under risk-neutral measure

Derivative Price $V_t$



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A linear direct mapping from Fourier Spectral Space

FT

**LEWIS REPRESENTATION**



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## The Lewis Standard Machine

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under risk-neutral measure

A linear direct mapping from Fourier Spectral Space

FT

$$V_t = \frac{e^{-r(T-t)}}{2\pi} \int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K} \phi_T(-z) \int_{-\infty}^{+\infty} e^{-izx} w(x)dxdz$$



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## The Lewis Standard Machine

Knowing $\widetilde{w}(z)$



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## The Lewis Standard Machine

Knowing $\widetilde{w}(z)$

implies reducing the problem to the calculation of a single integral

## The Lewis Standard Machine

$z = \xi + i\alpha$



## The Lewis Standard Machine

$z = \xi + i\alpha$

| Financial Claim | $w(x)$ | $\tilde{w}(x)$ |
|---|---|---|
| Call Option | $\max[S_T - K, 0]$ | $-\frac{K^{iz+1}}{z^2 - iz}, \ \alpha > 1$ |
| Put Option | $\max[K - S_T, 0]$ | $-\frac{K^{iz+1}}{z^2 - iz}, \ \alpha < 0$ |
| Covered Call | $\min[S_T, K]$ | $\frac{K^{iz+1}}{z^2 - iz}, \ 0 < \alpha < 1$ |
| Money Market | 1 | $2\pi\delta(k), \alpha \in \mathbb{R}$ |
| Self Quanto Call | $\max[S_T - K, 0] \cdot S_T$ | $\frac{K^{2+iz}}{(zi+1)^{S_T}(zi+2)^{S_T}}, \ \alpha < -2$ |
| Power Call | $\max[S_T - K, 0]^d$ | $\frac{K^{d(1+iz)}\Gamma(iz)\Gamma(d+1)}{\Gamma(iz+d+1)}, \alpha < -d$ |



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## DFT Convergence to FT

Given the General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad where \ m = 1,2,\ldots M$$



## DFT Convergence to FT

Given the General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad where \ m = 1,2,\ldots M$$

$$M \neq N$$



## DFT Convergence to FT

The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N\to\infty} \sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j, X)$$

$$t_m = \frac{2\pi}{X}(m-1)$$



## DFT Convergence to FT

$C_0$ via FT

Convergence theorems

$\overrightarrow{C_0}$ via DFT



## Syllabus of the presentation

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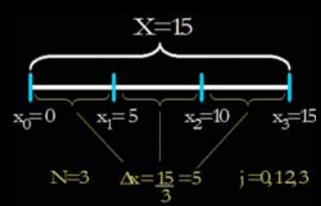
## Convergence Theorems for Uniform Grids

Condition 1

**Uniform Discretization Grid**



Condition 2

**N=M**

DFT specialized

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(n-1)} f(x_j) \quad \text{where } n=1,2,\ldots,N$$





Condition 1 Condition 2

**DFT Simplified Formula**

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i2\pi kj\gamma} f(x_j) \quad where \quad n = 1\ldots N$$





**Nyquist – Shannon Limit (N-S)**

$$\mathcal{F}[f(x)](t_n) = \lim_{N\to\infty} \frac{X}{N}\omega(n)$$

$\{t_n\}_{n=1..\frac{N}{2}}$ *for N even*

$\{t_n\}_{n=1..\frac{N+1}{2}}$ *for N odd*





$$C_t = \frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

**Uniform Discretization Grids for $\phi_T$**

1. $\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]$
2. $\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$





**1.**

$$C_t = \frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]$$

\+ N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_t - b + \lambda(u-1)]}}{b}\cdot\Re(\omega(u))$$





**2.**

$$C_t = \frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T(\xi_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$$

\+ N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_T - b + \lambda(u-1)]}}{3b}\cdot\Re(\omega(u))$$





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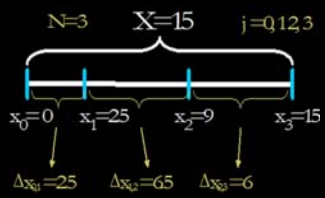
Condition 1

**Non Uniform Discretization Grid**








Condition 1

**Gaussian Grids**

Optimal choice of discretization points

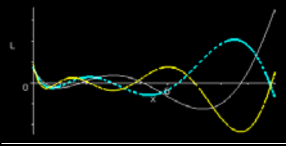
Gauss Laguerre Gander Gautschi

Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre



Zeros of
Laguerre Poynomials





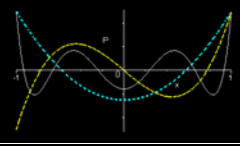
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Lobatto

Zeros of
Legendre Poynomials







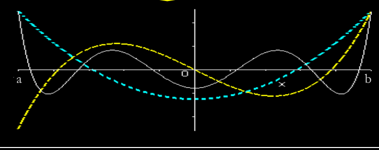
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gander Gautschi

Zeros of rescaled
Legendre Poynomials







Condition 2

N≠M

General DFT

$$\omega(m)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad \text{where } m=1,2,\ldots,2M$$





The Convergence Theorem
for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m)=\lim_{N\to\infty}\sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j,X)$$

$$t_m=\frac{2\pi}{X}(m-1)$$





$$C_t=\frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

Gaussian Grids for $\phi_T$

1. $\phi_T(\xi_{j-1})=e^{\left[1+i\left(\frac{M\pi}{a^*}-\ln S_t\right)\right]\xi_{j-1}}\Psi_0[\xi_{j-1}]\cdot\frac{1}{L_{N+1}(\xi_{j-1})L'_N(\xi_{j-1})}$

2. $\phi_T\left(\frac{1}{2}a(1+\xi_{j-1})\right)=e^{\left[-i\left(\frac{1}{2}a(1+\xi_{j-1})\right)\left(\ln S_t-\frac{M\pi}{a^*}\right)\right]}\Psi_0\left[\frac{1}{2}a(1+\xi_{j-1})\right]\cdot\frac{1}{\left[P_{N-1}(\xi_{j-1})\right]^2}$





1.

$$C_t=\frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T(\xi_{j-1})=e^{\left[1+i\left(\frac{M\pi}{a^*}-\ln S_t\right)\right]\xi_{j-1}}\Psi_0[\xi_{j-1}]\cdot\frac{1}{L_{N+1}(\xi_{j-1})L'_N(\xi_{j-1})}$$

\+

C-Th

$$C_0([\ln K]_u^*)\approx-\Re\left[\frac{e^{-a\left(\ln S_T-\frac{M\pi}{a^*}+\frac{2\pi}{a^*}(u-1)\right)}}{\pi}\frac{1}{N+1}\cdot\omega^*(u)\right]$$





2.

$$C_t=\frac{Ke^{-r(T-t)}}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-iz\ln K}\frac{\phi_T(-z)}{z^2-iz}dz$$

$$\phi_T\left(\frac{1}{2}a(1+\xi_{j-1})\right)=e^{\left[-i\left(\frac{1}{2}a(1+\xi_{j-1})\right)\left(\ln S_t-\frac{M\pi}{a^*}\right)\right]}\Psi_0\left[\frac{1}{2}a(1+\xi_{j-1})\right]\cdot\frac{1}{\left[P_{N-1}(\xi_{j-1})\right]^2}$$

\+

C-Th

$$C_0([\ln K]_u^*)\approx\Re\left[\frac{e^{-a\left(\ln S_T-\frac{M\pi}{a^*}+\frac{2\pi}{a^*}(u-1)\right)}}{\pi}\frac{1}{M(N-1)}\cdot\omega^*\left(\tfrac{1}{2}a(1+\upsilon_{j-1})\right)\right]$$





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## $\overrightarrow{C_t}$ via DFT

allows

Fast Fourier Trasform Algorithms





## $\overrightarrow{C_t}$ via DFT

Newton-Cotes

Fractional FFT





## $\overrightarrow{C_t}$ via DFT

Gauss

NonUniform FFT





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## Bayley-Swarztrauber F-DFT Characterization





## Bayley-Swarztrauber F-DFT Characterization

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i2\pi kj\gamma} f(x_j) \quad where \quad n = 1 \ldots N$$





## Bayley-Swarztrauber F-DFT Characterization

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i2\pi kj\gamma} f(x_j) \quad where \quad n = 1 \ldots N$$

with $\gamma$ that can be any complex number





If $\gamma = \frac{1}{N}$

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-1)} f(x_j) \quad where\ n = 1, 2, \ldots N$$

**The standard DFT definition**





## Choosing two indipendent uniform grids

## Choosing two indipendent uniform grids

$$x_j = jg\left(\tfrac{a}{N}\right) \quad for \quad j = 1 \ldots N$$ Spectral Grid

$$[\ln K]_u^{*} = \ln S_t - b + \lambda_u \quad for \quad u = 1, \ldots, N$$ Log-Strike Grid





## Choosing two indipendent uniform grids





## Choosing two indipendent uniform grids

Implies choosing a specific value of $\gamma$





## Choosing two indipendent uniform grids

Implies choosing a specific value of $\gamma$

$$\gamma = \frac{\lambda g\left(\frac{a}{N}\right)}{2\pi}$$





## Fast Fractional Reconstruction





## Fast Fractional Reconstruction

Step 1





## Fast Fractional Reconstruction

Step 1

Calculate sequences of $2N$ points





## Fast Fractional Reconstruction

Step 1

$$y = \left\{ \left( f\left((j-1)g\left(\tfrac{a}{N}\right)\right) e^{-i\pi j^2 \gamma} \right)_{j=0}^{N-1}, (0)_{j=0}^{N-1} \right\}$$

$$z = \left\{ \left( e^{i\pi j^2 \gamma} \right)_{j=0}^{N-1}, \left( e^{i\pi (N-j)^2 \gamma} \right)_{j=0}^{N-1} \right\}$$





## Fast Fractional Reconstruction

Step 2

Calculate

$$w = \psi_0\left((j-1)g\left(\tfrac{a}{N}\right)\right) \odot \left( e^{i\pi (N-j)^2 \gamma} \right)_{j=0}^{N-1}$$

Fast Fractional Reconstruction

**Step 3**

Calculate via standard FFT

$$\overline{w} = FFT(w), \overline{z} = FFT(z)$$





Fast Fractional Reconstruction

**Step 4**

Calculate

$$q = \overline{w} \odot \overline{z}$$





Fast Fractional Reconstruction

**Step 4**

Calculate

$$\omega(n) = IFFT(q) \oslash \left(e^{i\pi j^2 \gamma}\right)_{j=0}^{N-1}$$





Fast Fractional Reconstruction

The total computational cost drops

from

$$O(N^2)$$

to

$$O(6N\log_2 N)$$





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## Gaussian Gridding





Gaussian Gridding

**Step 1**





Gaussian Gridding

**Step 1**

Gaussian Convolution of the non uniformly sampled characteristic function





Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$

## Gaussian Gridding

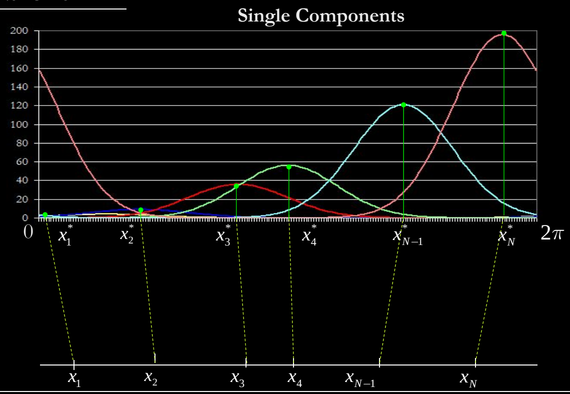
Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$












## Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

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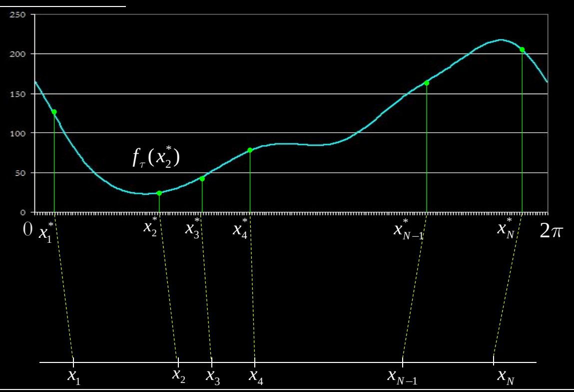







## Gaussian Gridding

Step 2

Discretization on an uniform oversampled grid of $f_\tau(x)$

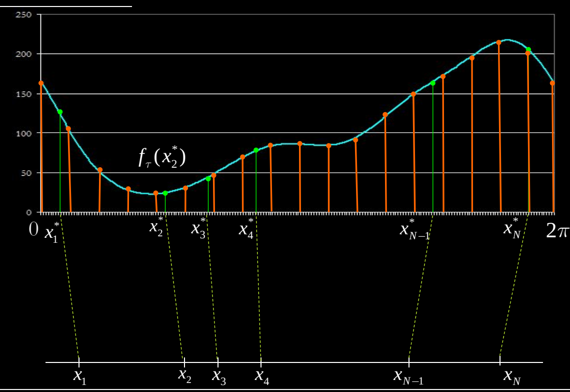
$$\tilde{f}_\tau(y_m) = \sum_{j=0}^{N-1} f(x_j)\, \tilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi)$$

where

$$\tilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi) = \begin{cases} \sum_{k=-\infty}^{\infty} e^{-\frac{(\ldots)^2}{4\tau}} \\ \text{oppure} \\ \sum_{k=-\infty}^{\infty} e^{-\frac{(\ldots)^2}{4\tau}} \end{cases}$$












## Gaussian Gridding

Step 3

Computation of the Fourier Coefficient of $f_\tau(x)$ discretised

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \sum_{m=0}^{M_\tau - 1} \tilde{f}_\tau\left(m \frac{2\pi}{M_\tau}\right) e^{-im\frac{2\pi}{M_\tau}(n-1)}$$





## Gaussian Gridding

Step 4

NU-DFT representation of the Fourier Coefficient $F_\tau(n)$

$$\tilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2 \tau} F_\tau(n)$$





## Gaussian Gridding

Step 5

DFT representation of the Fourier Coefficient $F_\tau(n)$

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, \ldots, \frac{M_\tau}{2}$

## Gaussian Gridding

Step 6

NU-DFT derivation as a function of DFT





## Gaussian Gridding

Step 6

NU-DFT derivation as a function of DFT

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

$$\widetilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2 \tau} F_\tau(n)$$

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, ...., \frac{M_\tau}{2}$





## Gaussian Gridding

Step 7

NU-FFT computation





## Gaussian Gridding

Step 7

NU-FFT computation

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

FFT





## Gaussian Gridding

Step 7

NU-FFT computation

$$\widetilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

NU-FFT ---------- FFT





## Computational Cost

The major computational cost of the Procedure is the FFT on the oversampled grid





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The major computational cost of the Procedure is the FFT on the oversampled grid

Choosing the oversampling ratio

$M_\tau = 2M$





## Computational Cost

The major computational cost of the Procedure is the FFT on the oversampled grid

Choosing the oversampling ratio

$M_\tau = 2M$

The total cost of the procedure is $\simeq 2M \log 2M$





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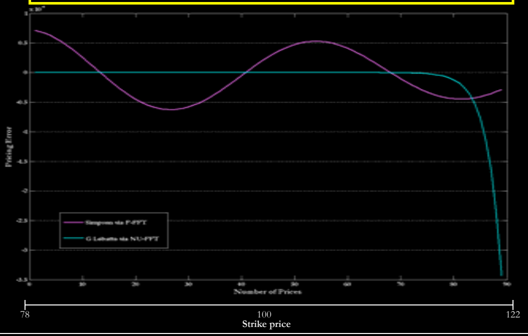
# ACCURACY





$\sigma = 0.3$

## BLACK SCHOLES MODEL

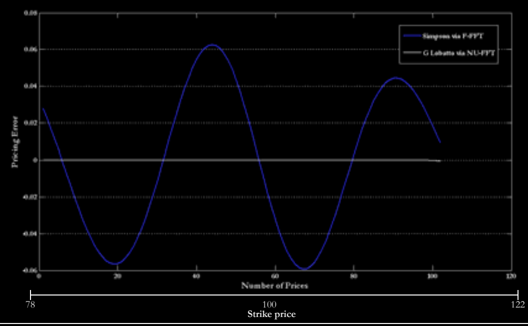







$\sigma = 0.1$

## BLACK SCHOLES MODEL





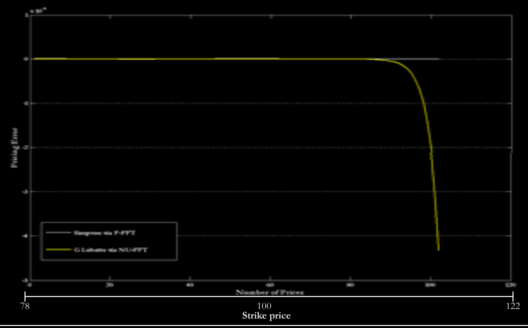



# STABILITY





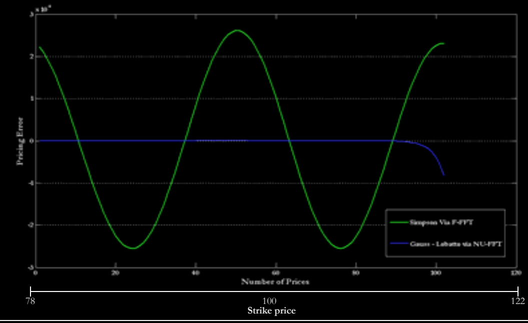
## BLACK SCHOLES MODEL







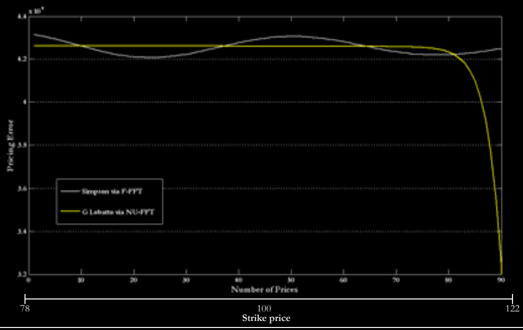

## HESTON MODEL








## MERTON MODEL



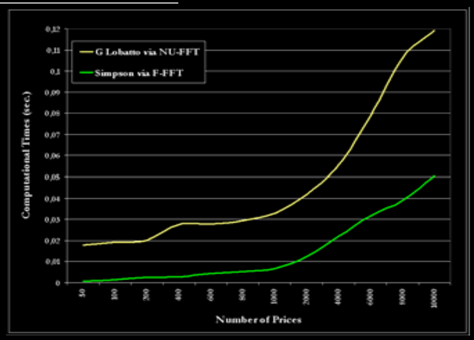




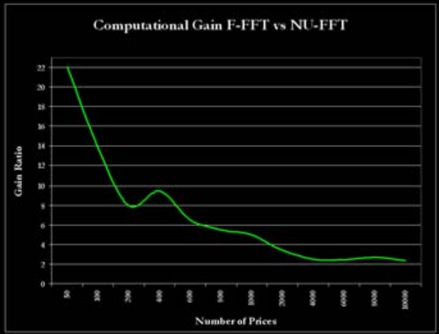

# SPEED








Centrino 1600Mhz – 1gb RAM
Mean Value over 1000 runs




Centrino 1600Mhz – 1gb RAM
Mean Value over 1000 runs





At very low time scales, the differences are negligible





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## Use of Gaussian Grids

| | |
|---|---|
| F-FFT | NO |
| NU – FFT | YES |





## Indifference to Nyquist-Shannon Limit

| | |
|---|---|
| F-FFT | YES |
| NU – FFT | YES |





## Indipendent Price Grids

| | |
|---|---|
| F-FFT | YES |
| NU – FFT | YES |





## FFT's like - Accuracy

| | |
|---|---|
| F-FFT | YES |
| NU – FFT | YES |





## Stability of Pricing

| | |
|---|---|
| F-FFT | NO |
| NU – FFT | YES |





## Speed of Pricing

| | |
|---|---|
| F-FFT | YES |
| NU – FFT | YES |

| | F-FFT | NU – FFT |
|---|---|---|
| Gaussian Grids | | ■ |
| NS Limit | ■ | ■ |
| Indipendent Grids | ■ | ■ |
| Accuracy | ■ | ■ |
| Stability | | ■ |
| Speed | ■ | ■ |