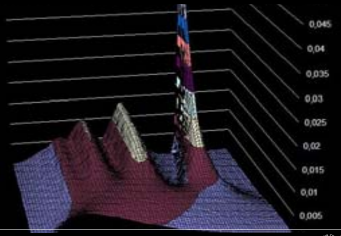
Sulflqj#Vwuxfwxuhg#Surgxfw#

Fourier Transform vs Discrete Fourier Trasform

From Theory to Trading Desk



Marcello Minenna - Paolo Verzella
Structured Products Europe 2006 - London

Syllabus of the presentation

- Unbundling structured products
  - Structured Bonds Italian market
- Review of Option Pricing in Affine Jump Diffusion Models
- Implementation of Fourier Trasform approach
- Implementation of Fast Fourier Trasform approach
  - Rules of Thumb

Syllabus of the presentation

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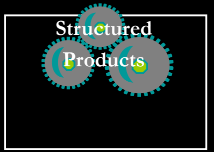
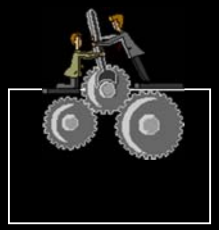
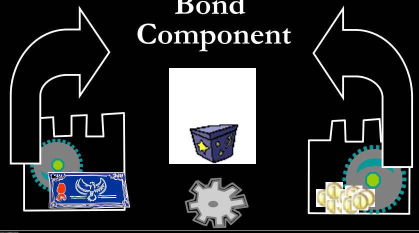
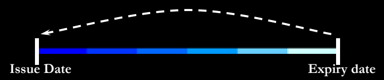




Bond Component



Present Value



Derivative Component

Derivative Component

REDUCE

Bond Component

Derivative Component

REDUCE

Bond Component

Structured Products

Derivative Component

INCREASE

Bond Component

Derivative Component

INCREASE

Bond Component

Structured Bonds

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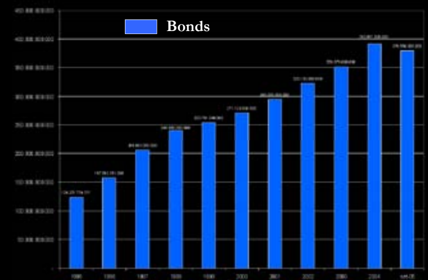
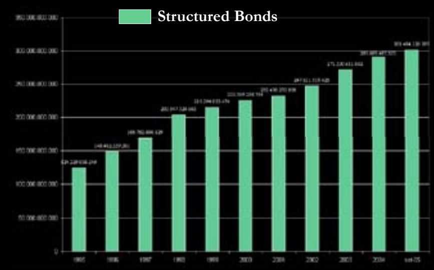
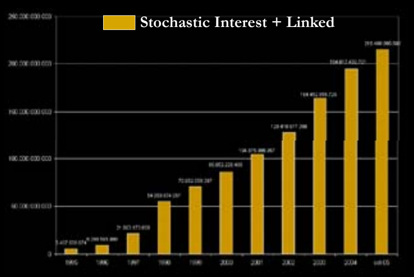
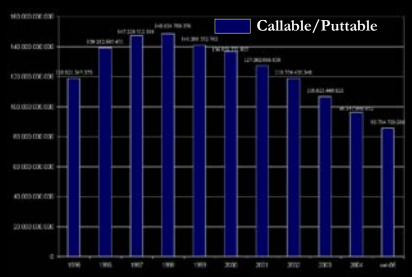
Structured Bonds in Italy

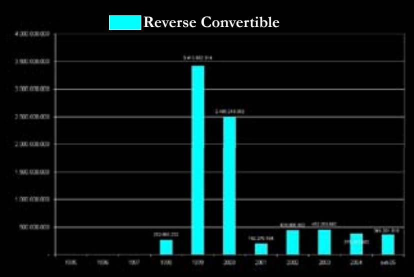
Primary Market trend
1995 – sept '05

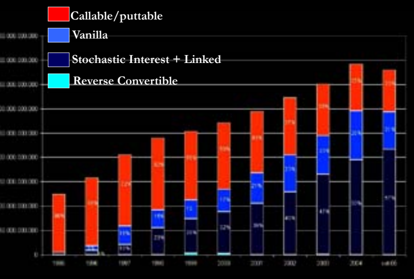


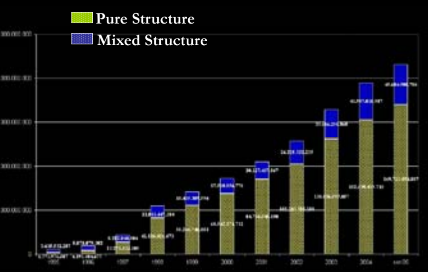












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Minenna, A guide to quantitative Finance, RiskBooks 2006

European Call Price C_t

Spot Price S_t

$P_1(\Theta), P_2(\Theta) = \Pr(\ln S_T \geq \ln K)$

under different martingale measures

European Call Price C_t

Spot Price S_t

$P_1(\Theta), P_2(\Theta) = \Pr(\ln S_T \geq \ln K)$

under different martingale measures

AJD models compute C_t

European Call Price C_t

Spot Price S_t

$P_1(\Theta), P_2(\Theta) = \Pr(\ln S_T \geq \ln K)$

under different martingale measures

AJD models compute C_t

PDE

$C_t = h[P_1(\Theta), P_2(\Theta)]$

European Call Price C_t

Spot Price S_t

$P_1(\Theta), P_2(\Theta) = \Pr(\ln S_T \geq \ln K)$

under different martingale measures

AJD models compute C_t

PDE

$C_t = h[P_1(\Theta), P_2(\Theta)]$

FT

$C_t = u\{g^{-1}[P_2(\Theta,\alpha)]\}$

European Call Price C_t

Spot Price S_t

$f_j(\ln S_T, \xi \mid \ln S_t) = \int_{-\infty}^{+\infty} e^{i\xi \ln S_T} q_j(\ln S_T \mid \ln S_t) d\ln S_T$

under different martingale measures

AJD models compute C_t

PDE

$C_t = h[P_1(f_1), P_2(f_2)]$

FT

$C_t = u(f_2)$

European Call Price C_t

Spot Price S_t

$f_2(\ln S_T, \xi \mid \ln S_t) = \int_{-\infty}^{+\infty} e^{i\xi \ln S_T} q(\ln S_T \mid \ln S_t) d\ln S_T$

Risk-neutral measure

AJD models compute C_t

PDE

$C_t = h[P_1(\Lambda_{1,2} f_2), P_2(f_2)]$

$\Lambda_{1,2} = \frac{S_T}{e^{r(T-t)} S_t}$

FT

$C_t = u(f_2)$

$$C_t = S_t\left[\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \Re\left(\frac{e^{i\xi \ln S_T}}{i\xi} f_1\right) d\xi\right] - Ke^{-r(T-t)}\left[\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \Re\left(\frac{e^{i\xi \ln S_T}}{i\xi} f_2\right) d\xi\right]$$

PDE

AJD models compute C_t

AJD models compute C_t

FT

$$C_t = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left(e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi}\right) d\xi$$

AJD models compute C_t

PDE

$C_t = h[P_1(f_1), P_2(f_2)]$

FT

$C_t = u(f_2)$

Numerical evaluation

Newton-Cotes

Hybrid

Gauss

AJD models compute C_t

PDE

$C_t = h[P_1(f_1), P_2(f_2)]$

FT

$C_t = u(f_2)$

Numerical evaluation

Newton-Cotes

Hybrid

Gauss

Newton-Cotes schemes compute C_t
Trapezoid rule

$$C_t \approx \left(\frac{1}{2}+\frac{1}{\pi}\right)\left[S_t\sum_{j=0}^{N}\frac{1}{2}a_j^N\cdot\Re\left(\frac{e^{-ij\frac{a}{N}\ln K}}{i\left(j+\varepsilon\delta_j\right)}f_1\left(j\frac{a}{N}\right)\right)-Ke^{-r(T-t)}\sum_{j=0}^{N}\frac{1}{2}a_j^N\cdot\Re\left(\frac{e^{-ij\frac{a}{N}\ln K}}{i\left(j+\varepsilon\delta_j\right)}f_2\left(j\frac{a}{N}\right)\right)\right]$$

PDE

Newton-Cotes schemes compute C_t
Trapezoid rule

Newton-Cotes schemes compute C_t
Trapezoid rule

FT

$$C_t \approx \frac{e^{-\alpha\ln K}}{2\pi}\frac{a}{N}\Re\left[\sum_{j=0}^{N}e^{-ijh\ln K}f_2\left(j\frac{a}{N}\right)\right]$$

Newton-Cotes schemes compute C_t
Simpson rule

$$C_t \approx \left(\frac{1}{2}+\frac{1}{\pi}\right)\left[S_t\sum_{j=0}^{N}\left[3+(-1)^{j+1}-\delta_j\right]\cdot\Re\left(\frac{e^{-ij\frac{a}{N}\ln K}}{i\left(j+\varepsilon\delta_j\right)}f_1\left(j\frac{a}{N}\right)\right)-\right.$$
$$\left.-Ke^{-r(T-t)}\sum_{j=0}^{N}\left[3+(-1)^{j+1}-\delta_j\right]\cdot\Re\left(\frac{e^{-ij\frac{a}{N}\ln K}}{i\left(j+\varepsilon\delta_j\right)}f_2\left(j\frac{a}{N}\right)\right)\right]$$

PDE

Newton-Cotes schemes compute C_t
Simpson rule

Newton-Cotes schemes compute C_t
Simpson rule

FT

$$C_t \approx \frac{e^{-\alpha\ln K}}{3\pi}\frac{a}{N}\Re\left[\sum_{j=0}^{N}\left[3+(-1)^{j+1}-\delta_j\right]\cdot e^{-ijh\ln K}f_2\left(j\frac{a}{N}\right)\right]$$

AJD models compute C_t

PDE: $C_t = h\left[P_1(f_1), P_2(f_2)\right]$

FT: $C_t = u\left(f_2\right)$

Numerical evaluation

Newton-Cotes

Hybrid

Gauss

Gauss schemes compute C_t
Gauss-Lobatto rule

$$C_t \approx S_t\left(\frac{1}{2}+\frac{1}{\pi}\right)a\left[\frac{1}{N(N-1)}\Re\left(\frac{e^{-i\varepsilon\ln K}}{i\varepsilon}f_1(\varepsilon)\right)+\Re\left(\frac{e^{-ia\ln K}}{ia}f_1(a)\right)+\right.$$
$$\left.+\sum_{j=2}^{N-1}\frac{1}{N(N-1)\left[P_{N-1}(\xi_j)\right]^2}\cdot\Re\left(\frac{e^{-i\left(\frac{1}{2}a\left(1+\xi_j\right)\right)\ln K}}{i\left(\frac{1}{2}a\left(1+\xi_j\right)\right)}f_1\left(\frac{1}{2}a\left(1+\xi_j\right)\right)\right)\right]-$$
$$-Ke^{-r(T-t)}\left(\frac{1}{2}+\frac{1}{\pi}\right)a\left[\frac{1}{N(N-1)}\Re\left(\frac{e^{-i\varepsilon\ln K}}{i\varepsilon}f_2(\varepsilon)\right)+\Re\left(\frac{e^{-ia\ln K}}{ia}f_2(a)\right)+\right.$$
$$\left.+\sum_{j=2}^{N-1}\frac{1}{N(N-1)\left[P_{N-1}(\xi_j)\right]^2}\cdot\Re\left(\frac{e^{-i\left(\frac{1}{2}a\left(1+\xi_j\right)\right)\ln K}}{i\left(\frac{1}{2}a\left(1+\xi_j\right)\right)}f_2\left(\frac{1}{2}a\left(1+\xi_j\right)\right)\right)\right]$$

PDE

Gauss schemes compute C_t
Gauss-Lobatto rule

Gauss schemes compute C_t
Gauss-Lobatto rule

FT

$$C_t \approx \frac{e^{-\alpha\ln K}}{\pi}a\left[\frac{1}{N(N-1)}\Re\left(f_2(0)\right)+\Re\left(e^{-ia\ln K}f_2(a)\right)+\right.$$
$$\left.+\sum_{j=2}^{N-1}\frac{1}{N(N-1)\left[P_{N-1}(\xi_j)\right]^2}\cdot\Re\left(e^{-i\left(\frac{1}{2}a\left(1+\xi_j\right)\right)\ln K}f_2\left(\frac{1}{2}a\left(1+\xi_j\right)\right)\right)\right]$$

$$C_t \approx -S_t\left(\frac{1}{2N}+\frac{1}{N\pi}\right)\sum_{j=1}^{N}\frac{1}{L_{N-1}\left(\xi_j\right)L'_N\left(\xi_j\right)}\Re\left(\frac{e^{-\xi_j(i\ln K-1)}}{i\xi_j}f_1\left(\xi_j\right)\right)+$$
$$+Ke^{-r(T-t)}\left(\frac{1}{2N}+\frac{1}{N\pi}\right)\sum_{j=1}^{N}\frac{1}{L_{N-1}\left(\xi_j\right)L'_N\left(\xi_j\right)}\Re\left(\frac{e^{-\xi_j(i\ln K-1)}}{i\xi_j}f_2\left(\xi_j\right)\right)$$

PDE

Gauss schemes compute C_t
Gauss-Laguerre rule

Gauss schemes compute C_t
Gauss-Laguerre rule

FT

$$C_t \approx -\frac{e^{-\alpha\ln K}}{N\pi}\sum_{j=1}^{N}\frac{1}{L_{N-1}(\xi_j)L'_{N-1}(\xi_j)}\cdot\Re\left(e^{-\xi_j(i\ln K-1)}f_2\left(\xi_j\right)\right)$$

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C_t via FT

$C_t = u\left\{g^{-1}\left[P_2(\Theta,\alpha)\right]\right\}$

calibration of α

by minimizing

C_t via PDE: $C_t = h\left[P_1(\Theta), P_2(\Theta)\right]$

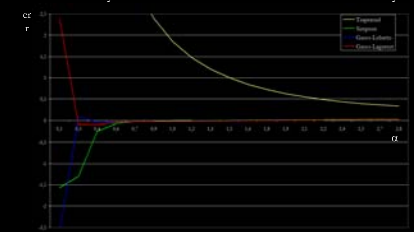
C_t via FT: $C_t = u\left\{g^{-1}\left[P_2(\Theta,\alpha)\right]\right\}$

calibration of α

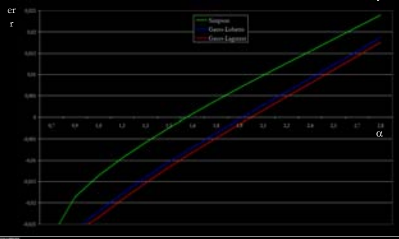
C_t via FT - spanning Θ, α

Stochastic volatility models

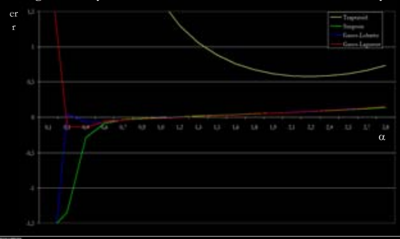
Low volatility — Short maturity



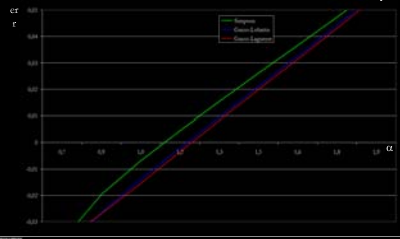
C_t via FT - spanning Θ, α
Stochastic volatility models
Low volatility zoom Short maturity



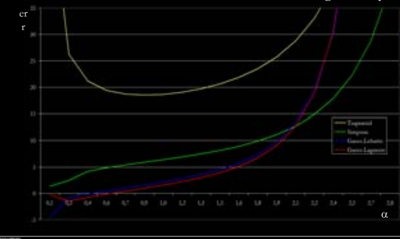
C_t via FT - spanning Θ, α
Stochastic volatility models
High volatility Short maturity



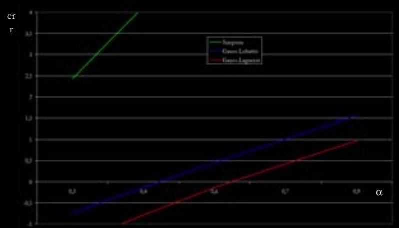
C_t via FT - spanning Θ, α
Stochastic volatility models
High volatility zoom Short maturity



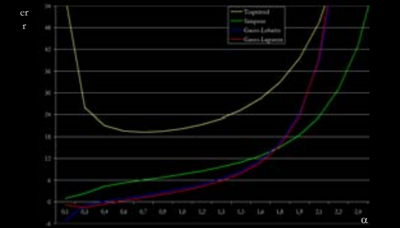
C_t via FT - spanning Θ, α
Stochastic volatility models
Low volatility Long maturity



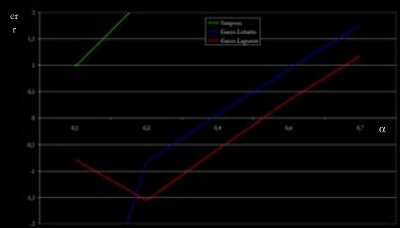
C_t via FT - spanning Θ, α
Stochastic volatility models
Low volatility zoom Long maturity



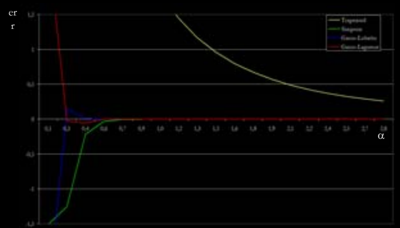
C_t via FT - spanning Θ, α
Stochastic volatility models
High volatility Long maturity



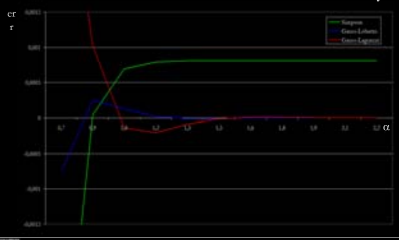
C_t via FT - spanning Θ, α
Stochastic volatility models
High volatility zoom Long maturity



C_t via FT - spanning Θ, α
Jump Diffusion models
Low volatility Short maturity



C_t via FT - spanning Θ, α
Jump Diffusion models
Low volatility zoom Short maturity



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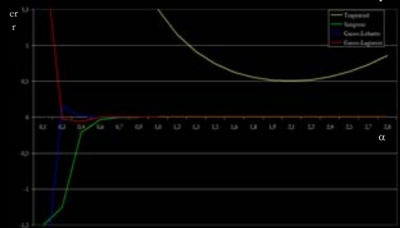

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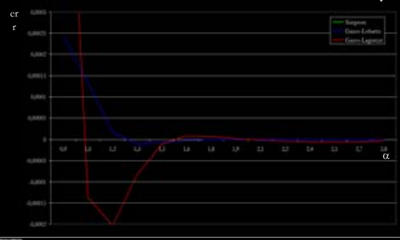

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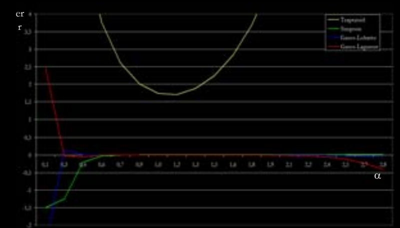
Low volatility Long maturity



C_t via FT - spanning Θ, α
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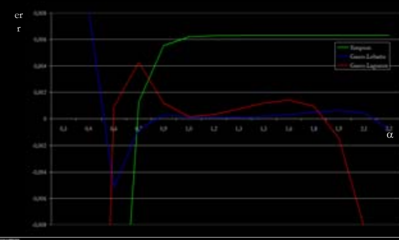
Low volatility zoom Long maturity



C_t via FT - spanning Θ, α
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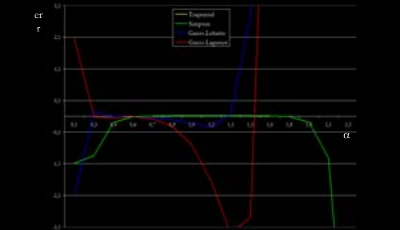

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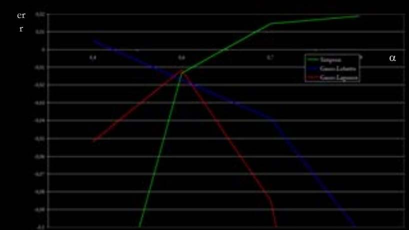

Option Pricing
requires
preliminary analysis of the option features

Option Pricing

requires

preliminary analysis of the option features

determines

choice of the pricing model

via PDE

C_t

Option Pricing

requires

preliminary analysis of the option features

determines

choice of the pricing model

via FT

Choice of α according to:
- Option features
- Quadrature algorithms

C_t

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C_t via FT

Convergence theorems

$\overrightarrow{C_t}$ via DFT

Minenna, A guide to quantitative Finance, RiskBooks 2006

C_t via FT

Convergence theorems

$\overrightarrow{C_t}$ via DFT

allows

Fast Fourier Trasform Algorithms

C_t via FT

Convergence theorems

$\overrightarrow{C_t}$ via DFT

Newton-Cotes

Cooley-Tuckey FFT

C_t via FT

Convergence theorems

$\overrightarrow{C_t}$ via DFT

Gauss

Fessler-Sutton NonUniform FFT

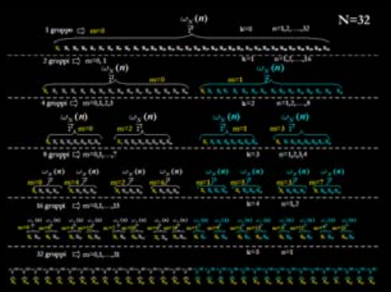
$$C_0\left(\left[\ln K\right]_u\right) \approx \frac{e^{-\alpha\left(\ln S_t - \frac{N\pi}{a} + \frac{2\pi}{a}(u-1)\right)}}{2\pi} \frac{a}{3N} \omega^*(u)$$

C-T FFT

Newton-Cotes Convergence Theorem

characterization compute $\overrightarrow{C_t}$

Simpson



Gauss Convergence Theorem

characterization compute $\overrightarrow{C_t}$

Gauss-Laguerre

F-S NU FFT

$$C_0\left(\left[\ln K\right]_u\right) \approx \frac{e^{-\alpha\left(\ln S_t - \frac{N\pi}{a} + \frac{2\pi}{a}(u-1)\right)}}{2\pi} \tilde{\omega}^*(\upsilon_u)$$

J. Fessler and B. P. Sutton, Non-uniform fast Fourier transforms using min-max interpolation. Signal Processing, 2001

C_t via FT

Convergence theorems

$\overrightarrow{C_t}$ via DFT

determines

Choice of α according to:
- Option features
- Quadrature algorithms

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C_t via FT - spanning Θ, α
Stochastic volatility models

VOLATILITY

MATURITY	LOW	HIGH
SHORT	Simpson [α≈(1,5;1,6)] Gauss-Lobatto [α≈(1,9;2)] Gauss-Laguerre [α≈(1,9;2)]	Simpson [α≈(1,05;1,1)] Gauss-Lobatto [α≈(1,15;1,2)] Gauss-Laguerre [α≈(1,15;1,2)]
LONG	Gauss-Lobatto [α≈(0,35;0,45)] Gauss-Laguerre [α≈(0,6;0,65)]	Gauss-Lobatto [α≈(0,4;0,45)] Gauss-Laguerre [α≈(0,5;0,55)]

C_t via FT - spanning Θ, α
Jump diffusion models

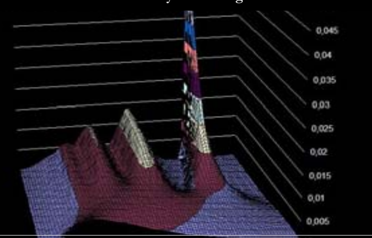
VOLATILITY

MATURITY	LOW	HIGH
SHORT	Gauss-Lobatto [α≈(1,4;2)] Gauss-Laguerre [α≈(1,55;2)]	Gauss-Lobatto [α≈(1,8;2,2)] Gauss-Laguerre [α≈(1,8;2,2)]
LONG	Gauss-Lobatto [α≈(0,9;1,2)] Gauss-Laguerre [α≈(0,95;1,05)]	Simpson [α≈(0,6;0,7)] Gauss-Lobatto [α≈(0,35;0,45)] Gauss-Laguerre [α≈(0,6)]

Sulflqj#Vwuxfwxuhg#Surgxfw#

Fourier Transform vs Discrete Fourier Trasform

From Theory to Trading Desk



Marcello Minenna - Paolo Verzella

Structured Products Europe 2006 - London