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A revisited and stable Fourier transform method for affine jump diffusion models

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Abstract

In the last decade, fast Fourier transform methods (i.e. FFT) have become the standard tool for pricing and hedging with affine jump diffusion models (i.e. AJD), despite the FFT theoretical framework is still in development and it is known that the early solutions have serious problems in terms of stability and accuracy. This fact depends from the relevant computational gain that the FFT approach offers with respect to the standard Fourier transform methods that make use of a canonical inverse Levy formula. In this work we revisit a classic FT method and find that changing the quadrature algorithm and using alternative, less flawed, representation for the pricing formulas can improve the computational performance up to levels that are only three time slower than FFT can achieve. This allows to have at the same time a reasonable computational speed and the well known stability and accuracy of canonical FT methods.

JEL classification: C63; C65; C69

Keywords: Option pricing; Fourier methods; Gauss lobatto; Quadrature schemes; Calibration performances

1. Introduction

Affine jump diffusion model are quite the most studied and well known Black–Scholes–Merton extensions. Their great analytical tractability and the presence of closed option pricing formulas make these models the ideal candidates to replace Black–Scholes–Merton in options pricing and hedging. One major issue is that the calibration to market prices of complex option pricing models with more parameters than a simple Black–Scholes volatility is still an open problem in financial mathematics. However, in the last decade, despite these well known aspects – specifically summarizable in the non-linearity, in the non-convexity of the optimization problem and in the numerical instability of pricing algorithms – the FFT approach has become the standard tool for pricing and hedging in the AJD context. This fact is due to the extraordinary gain in terms of computational speed that the FFT approach can achieve respect with the canonical inverse Levy formula developed at first in Heston (1993). The work of Carr and Madan (1999) makes possible for the first time the use of FFT algorithms in the context of option pricing. This method now allows to price and hedge in real time using models that are far beyond Black–Scholes–Merton, and a great number of empirical studies has adopted it enthusiastically. From our point of view, this massive shift towards the use of a innovative tool that is still in development, has greatly underestimated the problems, that are typical of a work in progress and not completely assessed, of stability and accuracy in pricing. In fact, few studies in literature have pointed out the accuracy problem inherent to Carr–Madan method, if applied just as it has been developed, for other option pricing model. In fact, their solution identifies a peculiar choice of the recombinant parameters that

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cannot be applied tout – court to all the models. A detailed discussion of the main issues involved can be found in Lee (2004). The classic Fourier transform method, on the other side, has not this type of problems and it can be greatly improved by simply changing to more accurate quadrature schemes and rewriting the pricing formula in order to avoid numerical instability problems. In this work the canonical method, as it appears in Heston, is analyzed in depth, rearranged with alternative formulations of the pricing formulas for the basic AJD models, and implemented with a new, robust Gauss–Lobatto quadrature scheme, as seen in Gander and Gautschi (2000). The results are an improved stability of the pricing formula and the calibration procedure, a great accuracy and a boosted speed of computation, far beyond than expected: in the better conditions, the Gauss–Lobatto scheme applied on the alternative formulas is only three times slower than FFT speed performances. Obviously, these computational times are not (yet) compatible with an operational use, but allow, for empirical studies, to have at the same time a reasonable computational speed and the well known stability and accuracy of canonical FT methods.

The work is structured in the following way: in the second and third section, the Black–Scholes–Merton model is considered as the simplest affine model, and the pricing formulas used in the empirical work, i.e. the formulas via the different Fourier transform methods (the FT canonical one, and FFT method as in Carr–Madan) are briefly presented, with the analytic connection with the classic formula.

The fourth section shows the classic AJD models (Heston, Merton, Bakshi–Cao–Chen) as natural generalizations of Black–Scholes–Merton model and derives, with details on the space transformations, the original and equivalent alternative formulas via FT.

Section 5 presents an analytical derivation of the AJD Greeks in the Black–Scholes–Merton style. The sixth section presents the numerical procedures tested and the empirical results in pricing, calibration and computation of the Greeks. Section 7 concludes.

2. The Black–Scholes–Merton model via Fourier transform

Theorem 1. *The Black–Scholes–Merton price of a CALL at the initial time $t = 0$, expressed in the form*

$$C_0 = S_0 N(d_1) - K\mathrm{e}^{-rT} N(d_2), \tag{1}$$

where

$$d_1 = \frac{\ln S_0 + (r + \frac{\sigma^2}{2})T - \ln K}{\sigma\sqrt{T}}, \tag{2}$$

$$d_2 = \frac{\ln S_0 + (r - \frac{\sigma^2}{2})T - \ln K}{\sigma\sqrt{T}} \tag{3}$$

and

$$N(d_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_k} \mathrm{e}^{-\frac{1}{2}u^2}\,\mathrm{d}u$$

has an equivalent representation in the form

$$C_0 = S_0 P_1 - K\mathrm{e}^{-rT} P_2,$$

where

$$P_k[x_{\tau=0} \geqslant \ln[K] | x_\tau] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{\mathrm{e}^{-i\xi\ln[K]}}{i\xi}\tilde{f}_k(x_{\tau=0}, \xi|x_\tau)\right]\mathrm{d}\xi \tag{4}$$

and

$$\tilde{f}_k(x_{\tau=0}, \xi|x_\tau) = \mathrm{e}^{C_\tau^{(k)} + i\xi x_\tau}$$

for $k = 1, 2$, with

$$C_\tau^{(1)} = ri\xi\tau + \frac{1}{2}\sigma^2[i\xi(i\xi+1)]\tau,$$
$$C_\tau^{(2)} = \frac{1}{2}\sigma^2 i\xi(i\xi-1)\tau + ri\xi\tau.$$

Proposition 2. *The equivalence between P_k and $N(d_k)$.*

The probability functions P_k in the form (4), for $k = 1, 2$ are equivalent representations of normal probability functions, i.e.

$$P_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_k} \mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z = N(d_k), \tag{5}$$

where d_k for $k = 1, 2$ are given by formulas (2), (3).

3. The Black–Scholes–Merton model via fast Fourier transform

Theorem 3. *Let the log-spot price risk neutral density be of the form*

$$q_T(\ln S_T) = \frac{1}{\sigma \ln S_T\sqrt{2\pi T}}\mathrm{e}^{-\frac{[\ln S_T - \ln S_0 - (r - \frac{\sigma^2}{2})T]^2}{2\sigma^2 T}},$$

i.e. the density function of a normally distributed random variable with mean $\ln S_0 + (r - \frac{\sigma^2}{2})T$ and standard deviation $\sigma\sqrt{T}$. Then let

$$\phi_T(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi\ln S_T} q_T(\ln S_T)\mathrm{d}\ln S_T \tag{6}$$

be the characteristic function (or Fourier transform) of this density. Under these conditions the Black–Scholes–Merton price of a CALL at the initial time $t = 0$, as expressed in Theorem 1 has an equivalent representation in the form[1]

[1] Carr and Madan (1999).

$$C_0(\ln K) = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{\infty} e^{-i\upsilon \ln K} \times \frac{e^{-rT}\phi_T(\upsilon - (\alpha+1)i)}{\alpha^2 + \alpha - \upsilon^2 + i(2\alpha+1)\upsilon} d\upsilon, \tag{7}$$

where

$$\phi_T(\upsilon - (\alpha+1)i) = e^{i(\upsilon-(\alpha+1)i)(\ln S_0 + rT - \frac{1}{2}\sigma^2 T) - \frac{1}{2}(\upsilon-(\alpha+1)i)^2\sigma^2 T}. \tag{8}$$

Proposition 4. [**Carr, Madan FFT representation**] *Given the definition of the discrete Fourier transform*

$$\omega(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j) \tag{9}$$

and making use of Trapezoid Rule, formula (7) can be approximated by the following expression:

$$C_0(\ln K_u) \simeq \frac{e^{-\alpha \ln K_u}}{\pi} \sum_{j=1}^{N} e^{-i\upsilon_j(-b+\lambda(u-1))} \psi_0(\upsilon_j)\eta, \tag{10}$$

where

$\eta = \frac{a}{N}$ *with a as the effective upper limit of integration,*
λ *as the regular spacing size of the log-strike space ranging*:

$$\ln(S_t) - \frac{N\lambda}{2} \leqslant \ln(S_t) \leqslant \ln(S_t) + \frac{N\lambda}{2}$$

and α *as the damping parameter.*

Notation 5. By now, we refer to α, η, λ as the Carr–Madan recombinant parameters.

Proposition 6. *Under the same hypotheses of Proposition 4 and making use of Simpson's Rule, the following alternative expression for Eq. (10) holds*

$$C_0(\ln K_u) \simeq \frac{e^{-\alpha \ln K_u}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)} e^{ib\upsilon_j} \psi_0(\upsilon_j) \times \frac{\eta}{3}(3 + (-1)^j - \delta_{j-1}), \tag{11}$$

where δ *is the Kronecker Delta Function.*

A FFT algorithm, as specified in Section (5), applies on (10) and (11) formulas.

4. Affine models via Fourier transform

The Fourier transform method used to derive the Black–Scholes–Merton FT pricing formula is now specified in more detail for the basic AJD models (i.e. Merton (1976) and Heston (1993) models). Several space transformations are used. Note that the pricing formula of the type a là Black–Scholes also appear in Geman et al. (1995) in relation to the choice of the numeraire tecnique.

4.1. The Merton model

The Merton model can be described with the following SDE:

$$dS_t = [r - \lambda\mu]S_t\,dt + \sqrt{v}S_t\,dW_t + S_tJ_t\,dq_t, \tag{12}$$

where

- $(q_t)_{t\geqslant 0}$ is a standard Poisson process with intensity $\lambda\,dt$, that is$\Pr(dq_t = 1) = \lambda\,dt$ and $\Pr(dq_t = 0) = 1 - \lambda\,dt$. $dq(t)$ is not correlated with J_t and $dW_t(W_t)_{t\geqslant 0}$. is a Standard Brownian Motion.
- J_t is the percentage jump size for the process $(S_t)_{t\geqslant 0}$, with probability distribution:

$$J_t \sim \mathrm{Log}N(\mu, (1+\mu)(e^{\sigma^2} - 1)) \tag{13}$$

- r is the instantaneous risk free rate at time t (constant).
- v is the variance of the process $(S_t)_{t\geqslant 0}$.

Theorem 7. *The risk-neutral partial differential equation for the replicating portfolio has the form*

$$-rf + \frac{\partial f}{\partial S}S[r - \lambda\mu] + \frac{\partial f}{\partial t} + \frac{1}{2}\left[\frac{\partial^2 f}{\partial S^2}[vS^2]\right] + \lambda E[f[(J+1)S, t] - f[S, t]] = 0 \tag{14}$$

also known as jump diffusion – Merton PDE.

4.1.1. PDE specification for the pricing of a Call option: derivation of Cauchy problem

Given the partial differential equation (14), the Cauchy problem – when f is exactly the price of a Call option C – is defined by PDE (14) specified to describe the Call price (15) and by its limit condition, that is, the pay-off value of the Call at expiration time T (a):

$$\underbrace{-rC + \frac{\partial C}{\partial S}rS + \frac{\partial C}{\partial t} + \frac{1}{2}\left[\frac{\partial^2 C}{\partial S^2}[vS^2]\right]}_{\text{diffusive component}} + \underbrace{+\lambda E_J[C[(J+1)S, t] - C[S, t]] - \frac{\partial C}{\partial S}\lambda S\mu}_{\text{Pure Jump component}} = 0, \tag{15}$$

$$C(S, t = T) = \mathrm{Max}[0, S - K], \tag{a}$$

where $C(S, t, T)$ is the CALL option price at time t.

4.1.2. The logarithmic version of Cauchy problem

The logarithmic version of Eq. (15) assumes the form

$$\underbrace{\frac{\partial C}{\partial t}+\frac{1}{2}\left[\frac{\partial^2 C}{\partial x^2}[v]\right]-rC+\frac{\partial C}{\partial x}\left(r-\frac{1}{2}v\right)}_{\text{diffusive component}} \tag{16}$$

$$\underbrace{+\lambda E\{C[x+\ln(J+1),t]-C[x,t]\}-\frac{\partial C}{\partial x}\lambda\mu}_{\text{Pure Jump component}}=0,$$

$$C(x,t=T)=\mathrm{Max}[0,\mathrm{e}^{x_T}-K]. \tag{a}$$

4.1.3. The shift in a pricing environment a-la Black–Scholes–Merton

Now the Cauchy problem (16) is specified in the classic Black–Scholes–Merton form with respective limit conditions, that is

$$C_t(S,t,T)=S_tP_1(S,v,t,T)-K\mathrm{e}^{-r(T-t)}P_2(S,v,t,T)$$
or $$(17)$$
$$C(x,t)=\mathrm{e}^xP_1(x,t)-K\mathrm{e}^{-r(T-t)}P_2(x,t)$$

$$\underbrace{\frac{\partial P_1}{\partial t}+\frac{1}{2}v\frac{\partial^2 P_1}{\partial x^2}+\left(r+\frac{1}{2}v\right)\frac{\partial P_1}{\partial x}}_{\text{diffusive component}}+$$

$$+\underbrace{\lambda E[[(J+1)P_1[x+\ln(J+1),t]-P_1(x,t)]-\lambda\mu\frac{\partial P_1}{\partial x}-\lambda\mu P_1(x,t)}_{\text{Pure Jump component}}$$

$$=0, \tag{18}$$

$$P_1(x,T)=1_{(x_T\geqslant\ln[K])}. \tag{a}$$

$$\underbrace{\frac{\partial P_2}{\partial t}+\frac{1}{2}v\left[\frac{\partial^2 P_2}{\partial x^2}\right]+\left(r-\frac{1}{2}v\right)\frac{\partial P_2}{\partial x}}_{\text{diffusive component}}$$

$$+\underbrace{\lambda E[[P_2[x+\ln(J+1),t]-P_2(x,t)]]-\lambda\mu\frac{\partial P_2}{\partial x}}_{\text{Pure Jump component}}=0, \tag{19}$$

$$P_2(x,T)=1_{(x_T\geqslant\ln[K])}. \tag{b}$$

Now, using the general Feyman–Kac formula for Levy processes (see Chan (1999)), a useful characterization of the P_k's probability measures at a generic time τ is given. In this case, for $k=1,2$, Feyman–Kac theorem is specified as follows:

$$P_k(x,t)=P_k[x_T\geqslant\ln[K]|x_t]. \tag{20}$$

4.1.4. The shift in Fourier space

The Cauchy Problem identified in (18), (19) is now shifted in a Fourier space, i.e. it is rearranged to be functionally dependent from the conditionally characteristic function of P_k's. It is known that

$$\begin{aligned}P_k(x_t)&=P_k[x_T\geqslant\ln[K]|x_t]\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-i\xi\ln[K]}}{i\xi}\tilde{f}_k(x_T,\xi|x_t)\mathrm{d}\xi\end{aligned} \tag{21}$$

so the following shifted PDE, with respective limit conditions, are derived

$$\begin{aligned}&-\lambda\mu\tilde{f}_1+\frac{\partial\tilde{f}_1}{\partial t}+\frac{1}{2}v\frac{\partial^2\tilde{f}_1}{\partial x^2}+\left[r-\lambda\mu+\frac{1}{2}v\right]\frac{\partial\tilde{f}_1}{\partial x}\\&\quad+\lambda E\left((1+J)\tilde{f}_1^{[J]}-\tilde{f}_1\right)\\&\quad=0,\end{aligned} \tag{22}$$

$$\tilde{f}_1(x_T,\xi)=\mathrm{e}^{i\xi[x_T]}, \tag{a}$$

$$\begin{aligned}&\frac{\partial\tilde{f}_2}{\partial t}+\frac{1}{2}v\frac{\partial^2\tilde{f}_2}{\partial x^2}+\left[r-\lambda\mu-\frac{1}{2}v\right]\frac{\partial\tilde{f}_2}{\partial x}+\lambda E\left(\tilde{f}_2^{[J]}-\tilde{f}_2\right)\\&\quad=0,\end{aligned} \tag{23}$$

$$\tilde{f}_2(x_T,\xi)=\mathrm{e}^{i\xi[x_T]}. \tag{b}$$

In order to get an explicit solution of Cauchy problem, note that a temporal shift in Fourier space is needed.

4.1.5. Specification of Cauchy problem as an ODEs system

Let us guess that the solution f_k of Cauchy problem (22), (23), takes the following form for $k=1,2$:

$$\tilde{f}_k(x_{\tau=0},\xi|x_\tau)=\mathrm{e}^{C_\tau^{(k)}+i\xi x_\tau}. \tag{24}$$

It is easy to note that (24) and time shifted (22), (23) imply that, for $\tau=0$, $C_{\tau=0}^{(k)}=0$. By making a wise use of the (24) form, we obtain

$$\begin{aligned}\frac{\partial C_t^{(1)}}{\partial\tau}&=-\lambda\mu(i\xi+1)+\frac{1}{2}vi\xi(\xi+1)+ri\xi\\&\quad+\lambda\left[\mathrm{e}^{\frac{\sigma^2}{2}(i\xi+1)}(1+\mu)^{(i\xi+1)}-1\right],\end{aligned} \tag{25}$$

$$C_{\tau=0}^{(1)}=0. \tag{a}$$

$$\begin{aligned}\frac{\partial C_\tau^{(2)}}{\partial\tau}&=\frac{1}{2}vi\xi(i\xi-1)+ri\xi-\lambda\mu i\xi\\&\quad+\lambda\left[\mathrm{e}^{\frac{\sigma^2}{2}i\xi(i\xi-1)}(1+\mu)^{i\xi}-1\right],\end{aligned} \tag{26}$$

$$C_{\tau=0}^{(2)}=0. \tag{b}$$

4.1.6. The Cauchy Problem solution

By solving the highlighted Cauchy problems, we get

$$\begin{aligned}C_\tau^{(1)}&=ri\xi\tau-\lambda\mu[i\xi+1]\tau+\frac{1}{2}v[i\xi(i\xi+1)]\tau+\\&\quad+\lambda\tau\mathrm{e}^{\frac{\sigma^2}{2}(i\xi+1)}(1+\mu)^{(i\xi+1)}-\lambda\tau,\end{aligned} \tag{27}$$

$$\begin{aligned}C_\tau^{(2)}&=\frac{1}{2}vi\xi\tau(i\xi-1)+ri\xi\tau-\lambda\mu i\xi\tau\\&\quad+\lambda\tau\mathrm{e}^{\frac{\sigma^2}{2}i\xi(i\xi-1)}(1+\mu)^{i\xi}-\lambda\tau.\end{aligned} \tag{28}$$

By sticking in (24) the highlighted values for $C_\tau^{(1)}$, $C_\tau^{(2)}$- given by (27), (28) – and then by using the resulting value of (24) in (21), we get the analytical formula for probability functions P_k that, adopted in expression (17), gives back the Call option price searched.

In the analyzed framework the following proposition holds:

Proposition 8. *There exists the following equivalent representation for the formula given in (21), i.e.*

$$P_k[x_{\tau=0} \geqslant \ln[K]|x_\tau] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\xi\ln[K]}}{i\xi}\tilde{f}_k(x_{\tau=0}, \xi|x_\tau)\right]d\xi. \tag{29}$$

Running Merton model requires specific routines for the quadrature of integral in (29).

4.2. The Heston model

This model is defined by the following system of stochastic differential equations (SDE):

$$dS_t = \mu S_t\,dt + \sqrt{v_t}S_t\,dz_t^{(1)}, \tag{30}$$

$$dv_t = \kappa[\theta - v_t]dt + \sigma\sqrt{v_t}\,dz_t^{(2)}, \tag{31}$$

where

- (30) is the SDE that describes the spot price dynamics,
- $(S_t)_{t\geqslant 0}$ is the spot price process,
- μ is the return log-rate,
- $(v_t)_{t\geqslant 0}$ is the variance process,
- $(z_t^{(1)})_{t\geqslant 0}$ is a standard Wiener process,
- (31) is the SDE that describes the dynamics of the variance process v_t,
- κ, θ, σ are constant parameters of SDE, (31),
- $(z_t^{(2)})_{t\geqslant 0}$, is a standard Wiener process,
- $\rho_{1,2}$ is the correlation coefficient between $z_t^{(1)}$ and $z_t^{(2)}$, that is $dz_t^{(1)} \cdot dz_t^{(2)} = \rho_{1,2}\,dt$.

In the following, the notation will be simplified and the subscript t will be omitted. So, from now: $dz_t^{(i)} = dz_i$.

Theorem 9. *The risk – neutral partial differential equation for the replicating portfolio has the form*

$$-rf + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}vS^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma^2 v + \frac{\partial f}{\partial S\partial v}S\sigma\rho_{1,2}v + \frac{\partial f}{\partial v}[\kappa(\theta - v) - \lambda^*(S,v,t)] = 0 \tag{32}$$

also known as Stochastic Volatility – Heston PDE. $\lambda^*(S,v,t)$ is also known as the premium for variance risk.

4.2.1. Specification of the PDE for the pricing of a Call option: derivation of Cauchy problem

Given the partial differential equation (32), the Cauchy problem – when f is exactly the price of a Call option C – is defined by PDE (32) specified to describe the Call price (33) and by its limit condition, that is, the pay-off value of the Call at expiration time T (a):

$$-rC + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}rS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}vS^2 + \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma^2 v + \frac{\partial C}{\partial S\partial v}S\sigma\rho_{1,2}v + \frac{\partial C}{\partial v}[\kappa(\theta - v) - \lambda^*(S,v,t)] = 0, \tag{33}$$

$$C(S_t, v_t, t = T) = \max(0, S_T - K), \tag{a}$$

where $C_t(S, v, t, T)$ is the CALL price at time t.

4.2.2. The shift of Cauchy problem in a forward space

Eq. (32), is transformed now in its equivalent version in terms of *forward* prices.

So defined the forward price $F_{t,T} \doteq e^{r(T-t)}S_t$, this gives:

$$-\frac{\partial\widetilde{C}}{\partial\tau} + r\frac{\partial\widetilde{C}}{\partial x} + \frac{1}{2}\frac{\partial^2\widetilde{C}}{\partial v^2}(\sigma^2 v) + \frac{\partial^2\widetilde{C}}{\partial x\partial v}(v\sigma\rho_{1,2}) + \frac{1}{2}\left(\frac{\partial^2\widetilde{C}}{\partial x^2} - \frac{\partial\widetilde{C}}{\partial x}\right)v + \frac{\partial\widetilde{C}}{\partial v}[\kappa(\theta - v) - \tilde{\lambda}v] = 0, \tag{34}$$

$$\widetilde{C}(x_\tau, v_\tau, \tau = 0) = \max(0, e^{x_{\tau=0}} - K). \tag{a}$$

4.2.3. The shift in a pricing environment a-la Black–Scholes–Merton

Now the Cauchy problem (34) is specified in the classic Black–Scholes–Merton form, with respective limit conditions that is

$$\begin{aligned} &C_t(S, v, t, T) = S_t P_1(S, v, t, T) - Ke^{-r(T-t)}P_2(S, v, t, T) \\ &\text{or } C_t(x, v, \tau) = S_t P_1(x, v, \tau) - Ke^{-r\tau}P_2(x, v, \tau) \end{aligned} \tag{35}$$

where P_1, P_2 are probability measures.

$$-\frac{\partial P_j}{\partial\tau} + \frac{\partial P_j}{\partial x}(r + c_j v) + \frac{1}{2}\frac{\partial^2 P_j}{\partial v^2}(\sigma^2 v) + \frac{\partial^2 P_j}{\partial x\partial v}(v\sigma\rho_{1,2}) + \frac{1}{2}\frac{\partial^2 P_j}{\partial x^2}v + \frac{\partial P_j}{\partial v}(a - b_j v) = 0, \tag{36}$$

$$P_j(x_\tau, v_\tau, \tau = 0) = 1_{(x_\tau \geqslant \ln K)}. \tag{a}$$

Now, using the Feyman–Kac formula, a useful characterization of the P_j's probability measures at a generic time τ is given. In this case, Feyman–Kac theorem is specified as follows:

$$P_j(x_\tau, v_\tau, \tau) = P_j(x_{\tau=0} \geqslant \ln K \ \ | x_\tau, v_\tau). \tag{37}$$

4.2.4. The Shift in Fourier Space

The Cauchy Problem identified in (36) is now shifted in a Fourier space, i.e. it is rearranged to be functionally dependent from the conditionally characteristic function of P_j's.

$$P_j(x_{\tau=0} \geqslant \ln K \ \ | x_\tau, v_\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\xi\ln K}}{i\xi}\tilde{f}_j(x_\tau, v_\tau, \tau = 0, \xi \mid \ x_\tau, v_\tau)\,d\xi \tag{38}$$

so, the following shifted PDE, with respective limit conditions, are derived

$$\frac{\partial \tilde{f}_j}{\partial \tau} + \frac{\partial \tilde{f}_j}{\partial x}(r + c_j v) + \frac{1}{2}\frac{\partial^2 \tilde{f}_j}{\partial v^2}(\sigma^2 v) + \frac{\partial^2 \tilde{f}_j}{\partial v \partial x}(v\sigma\rho_{1,2}) + \frac{\partial^2 \tilde{f}_j}{\partial x^2} v + \frac{\partial \tilde{f}_j}{\partial v}[a - b_j v] = 0, \tag{39}$$

$$\tilde{f}_j(x_\tau, v_\tau, \tau = 0, \xi) = e^{i\xi x_{\tau=0}}. \tag{a}$$

4.2.5. The specification of Cauchy Problem as an ODEs system

Let us guess that the solution $\tilde{f}_j$ of Cauchy problem (39), takes the form

$$\tilde{f}_j(x_\tau, v_\tau, \tau = 0, \xi | x_\tau, v_\tau) = e^{\left(C_\tau^{(j)} + D_\tau^{(j)} v_t + i\xi x_\tau\right)}. \tag{40}$$

It is easy to note that (40) and (a) imply that, for $\tau = 0$, $C_0^{(j)} = 0$ and $D_0^{(j)} = 0$. By making a wise use of the (40) form, we obtain

$$\frac{\partial C_j}{\partial \tau} = ri\xi + aD_j, \tag{41}$$

$$\frac{\partial D_j}{\partial \tau} = c_j i\xi + \frac{1}{2}D_j^2\sigma^2 + i\xi D_j \sigma\rho_{1,2} - \frac{1}{2}\xi^2 - b_j D_j, \tag{42}$$

$$C_0^{(j)} = 0, \tag{a}$$

$$D_0^{(j)} = 0, \tag{b}$$

that is, a system of ordinary differential equations (ODE) with respective final conditions.

4.2.6. The Cauchy Problem solution

By solving the highlighted Cauchy problems we get

$$D_\tau^{(j)} = -\frac{2\alpha_2}{\sigma^2}\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}, \tag{43}$$

$$C_\tau^{(j)} = ri\xi\tau - \frac{2a}{\sigma^2}\left(\alpha_2\tau + \ln\frac{ge^{d\tau} - 1}{g - 1}\right), \tag{44}$$

where

$$d = \sqrt{(\rho_{1,2}\sigma\xi i - b_j)^2 - \sigma^2(2c_j\xi i - \xi^2)}, \tag{45}$$

$$\alpha_1 = \frac{\rho_{1,2}\sigma\xi i - b_j + d}{2}, \alpha_2 = \frac{\rho_{1,2}\sigma\xi i - b_j - d}{2}, \tag{46}$$

By sticking in (40) the highlighted values for $C_\tau^{(j)}$, $D_\tau^{(j)}$ – given by (43), (44) – and then by using the resulting value of (40) in (38), we get the analytical formula for probability functions P_k that, adopted in expression (35), gives back the Call option price searched.

In the analyzed framework the following proposition holds:

Proposition 10. *There exists the following equivalent representation for the formula given in* (38), *i.e.*

$$P_j(x_{\tau=0} \geqslant \ln K \mid x_\tau, v_\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\xi \ln K}\tilde{f}_j(x_\tau, v_\tau, \tau = 0, \xi \mid x_\tau, v_\tau)}{i\xi}\right] d\xi. \tag{47}$$

Running Heston model requires specific routines for the quadrature of integral in (47).

Proposition 11. *Empirical Tests show that the below equivalent formula for $C_\tau^{(j)}$ improves pricing stability*:

$$C_j = ri\xi\tau - \frac{a}{\sigma^2}\left(\rho_{1,2}\sigma\xi i - b_j + d\right)\tau - \frac{a}{\sigma^2}2 \times \ln\left(1 - \frac{(1 - e^{-d\tau})(\rho_{1,2}\sigma\xi i - b_j + d)}{2d}\right). \tag{48}$$

Proposition 12. *Empirical Tests show that the below equivalent formula for $D_\tau^{(j)}$ improves pricing stability*:

$$D_j = \frac{(2c_j\xi i - \xi^2)(1 - e^{-d\tau})}{2d - (\rho_{1,2}\sigma\xi i - b_j + d)(1 - e^{-d\tau})} \tag{49}$$

Note that this formula is easily extended to more general AJD models (see Bakshi et al. (1997), Bakshi and Cao (2004)).

5. Computing the Greeks: The homogeneity approach

A similar approach for greeks has been used by Reiss and Wystup (2001). *The Greeks in Merton model*:

Notation 13. From now, this simplified notation is also used

$$\tilde{f}_j \doteq \tilde{f}_j(x_\tau, \tau = 0, \xi \mid x_\tau). \tag{50}$$

Proposition 14. *The following identities hold*:

$$\frac{\partial}{\partial S_t}\tilde{f}_j(x_\tau, \tau = 0, \xi \mid x_\tau) = \tilde{f}_j(x_\tau, \tau = 0, \xi | x_\tau) \cdot i\xi\frac{1}{S_t}, \tag{51}$$

$$\frac{\partial^2}{\partial S_t^2}\left(\tilde{f}_j(x_\tau, \tau = 0, \xi | x_\tau)\right) = \frac{1}{S_t^2}i\xi(i\xi - 1)\tilde{f}_j(x_\tau, \tau = 0, \xi | x_\tau). \tag{52}$$

Proposition 15. *The following identities hold*:

$$\frac{\partial}{\partial K}\left[\frac{e^{-i\xi \ln K}}{i\xi}\right] = -\frac{1}{K}e^{-i\xi \ln K}, \tag{53}$$

$$\frac{\partial^2}{\partial K^2}\left[\frac{e^{-i\xi \ln K}}{i\xi}\right] = \frac{1}{K^2}e^{-i\xi \ln K}(i\xi + 1). \tag{54}$$

Proposition 16. *The following identities hold*:

$$\frac{\partial P_j}{\partial S_t} = \frac{1}{\pi S_t}\int_0^\infty \Re[e^{-i\xi \ln K}(\tilde{f}_j(x_\tau, \tau = 0, \xi | x_\tau))]\, d\xi, \tag{55}$$

$$\frac{\partial^2 P_j}{\partial S_t^2} = \frac{1}{\pi S_t^2}\int_0^\infty \Re[e^{-i\xi \ln K}((i\xi + 1)\tilde{f}_j(x_\tau, \tau = 0, \xi | x_\tau))]\, d\xi, \tag{56}$$

$$\frac{\partial P_j}{\partial K} = -\frac{1}{\pi K}\int_0^\infty \Re[(e^{-i\xi \ln K}\tilde{f}_j(x_\tau, \tau = 0, \xi \mid x_\tau))]d\xi, \tag{57}$$

$$\frac{\partial^2 P_j}{\partial K^2} = \frac{1}{\pi K^2}\int_0^\infty \Re[e^{-i\xi \ln K}(i\xi + 1)\tilde{f}_j(x_\tau, \tau = 0, \xi \mid x_\tau)]d\xi. \tag{58}$$

Proposition 17. *The following identities hold*:

$$\frac{\partial P_1}{\partial v_t} = \frac{1}{\pi}\int_0^{\infty} \Re\left[\frac{\mathrm{e}^{-i\xi \ln K}}{i\xi}\left(\mathrm{e}^{C_\tau^{(1)}+i\xi x_\tau}\cdot\frac{1}{2}i\xi(i\xi+1)\tau\right)\right]\mathrm{d}\xi, \qquad (59)$$

$$\frac{\partial P_2}{\partial v_t} = \frac{1}{\pi}\int_0^{\infty} \Re\left[\frac{\mathrm{e}^{-i\xi \ln K}}{i\xi}\left(\mathrm{e}^{C_\tau^{(2)}+i\xi x_\tau}\cdot\frac{1}{2}i\xi(i\xi-1)\tau\right)\right]\mathrm{d}\xi. \qquad (60)$$

Proposition 18. *The following identities hold*:

$$\frac{\partial P_1}{\partial t} = \frac{1}{\pi}\int_0^{\infty} \Re\left[\frac{\mathrm{e}^{-i\xi \ln K}}{i\xi}\mathrm{e}^{C_\tau^{(1)}+i\xi x_\tau}\left(-ri\xi+\lambda\mu[i\xi+1] -\frac{1}{2}v[i\xi(i\xi+1)]-\lambda\mathrm{e}^{\frac{\sigma^2}{2}(i\xi+1)}(1+\mu)^{(i\xi+1)}+\lambda\right)\right]\mathrm{d}\xi, \qquad (61)$$

$$\frac{\partial P_2}{\partial t} = \frac{1}{\pi}\int_0^{\infty} \Re\left[\frac{\mathrm{e}^{-i\xi \ln K}}{i\xi}\mathrm{e}^{C_\tau^{(2)}+i\xi x_\tau}\left(-\frac{1}{2}vi\xi(i\xi-1)-ri\xi +\lambda\mu i\xi-\lambda\mathrm{e}^{\frac{\sigma^2}{2}i\xi(i\xi-1)}(1+\mu)^{i\xi}+\lambda\right)\right]\mathrm{d}\xi. \qquad (62)$$

Proposition 19. *The* **Delta** *of a CALL option is*

$$\Delta_C = P_1. \qquad (63)$$

Proposition 20. *The* **Gamma** *of a CALL option is*

$$\Gamma_C = \frac{\partial P_1}{\partial S_t} \qquad (64)$$

(*for* $\frac{\partial P_1}{\partial S_t}$, *cfr. expression* (55)).

Proposition 21. *The* **Vega of a** *CALL option is*

$$\mathscr{V}_C = S_t\frac{\partial P_1}{\partial v_t} - K\mathrm{e}^{-r\tau}\frac{\partial P_2}{\partial v_t} \qquad (65)$$

(*for* $\frac{\partial P_j}{\partial v_t}$, *cfr. expression* (59)).

Corollary 22. *The second order cross derivative respect to* v_t *and* S_t *is*

$$\frac{\partial^2 C_t}{\partial S_t \partial v_t} = \frac{\partial P_1}{\partial v_t}. \qquad (66)$$

Proposition 23. *The* **Theta** *of a CALL option is*

$$\Theta_C = S_t\frac{\partial P_1}{\partial t} - K\left(r\mathrm{e}^{-r(T-t)}P_2 + \mathrm{e}^{-r(T-t)}\frac{\partial P_2}{\partial t}\right) \qquad (67)$$

(*for* $\frac{\partial P_j}{\partial t}$, *cfr. expression* (61), (62)).

Proposition 24. *The* **Rho** *of a CALL option is*

$$\rho_C = K\tau\mathrm{e}^{-r\tau}P_2. \qquad (68)$$

The Greeks in Heston model: Very similar in the analytic form to those derived for Merton model.

6. Numerical procedures and results

The models tested are the canonical ones, i.e. Heston (1993), Merton (1976), Bakshi et al. (1997).

6.1. Pricing

Three different algorithms are used to have a numerical approximation of the integrands involved in the pricing formulas:

OLD FT-Q: An High Order (up to 8th) Newton–Cotes quadrature algorithm.

NEW FT-Q: An Adaptive and Iterative Gauss–Lobatto quadrature algorithm.

Definition 25. Gauss–Lobatto quadrature scheme

Let is define as Gauss–Lobatto quadrature scheme a Gaussian quadrature with weighting function $W(x) = 1$, in which the endpoints of the interval $[-1, 1]$ are included in a total of n abscissas, giving $r = n - 2$ free abscissas. Abscissas are symmetrical about the origin, and the general formula is

$$\int_{-1}^{1} f(x)dx = w_1 f(-1) + w_n f(1) + \sum_{i=2}^{n-1} w_i f(x_i),$$

where

$$w_i = \frac{2}{N(N-1)[P_{N-1}(x_i)]^2}$$

and

$$w_1 = w_N = \frac{2}{N(N-1)}$$

The free abscissas x_i for $i = 2, \ldots, n-1$ are the roots of the polynomial $P'_{n-1}(x)$ where $P(x)$ is a Legendre polynomial.[2]

FFT: A fast Fourier transform algorithm (Cooley–Tukey algorithm mixed with prime factor and split-radix algorithms).

Different Valuation Criteria are defined to assess the algorithms performance with respect to the pricing formulas.

Stability: The algorithm is defined stable if and only if it "closes" the quadrature scheme, giving a "reasonable" result different from a NaN value, after the pricing formula has been spanned on a vast area of the parameters set.

Accuracy: The algorithm is defined accurate with respect to the results of the NEW FT-Q algorithm.[3]

Speed: The algorithm is defined fast with respect to the results of the FFT algorithm on a set of 4100 prices along the strike, with fixed pricing formulas parameters.

The FFT algorithm is implemented using the following recombinant parameters: $\alpha = 1.45$ (the damping parame-

[2] For further details, see Hunter and Nikolov (2000).

[3] This criterion is related to the fact that the pricing of the Call price under the Black–Scholes–Merton model via Fourier transform adopting the Gauss–Lobatto quadrature algorithm gives back always exactly the Call price computed via standard Black–Scholes–Merton approach.

ter) and the standard values as they appear in Carr and Madan (1999) for the discretization grids (log-strike and characteristic function). When an equivalent pricing formula is available (see Propositions 11, 12), the test are performed again to assess the performance gains. For each run the following "market" parameters are used: $S_0 = 98$, $K = 100, \tau = 0.5$.¶PRICING PERFORMANCES: Tables for Heston model

With OR we are denoting the use of models original formulas
With AL we are denoting the use of models alternative formulas
NC = Not Computed (due to instability problems)

Heston model via OLD FT-Q

CPU times for pricing (sec.)

Param.	θ_v	κ_v	σ_v	λ	ρ	v	OR	AL
Case I	0.45	2	0.1	0.5	0	0.2	42.4	43.0
Case II	0.8	1	0.4	1.6	0.7	0.5	44.7	42.1
Case III	0.01	3	0.2	−1	−0.7	0.05	NC	56.6
Case IV	0.2	0.3	0.75	0	−0.1	0.2	52.0	48.5
Case V	0.3	2.2	0.15	−2	1	0.3	NC	NC
Case VI	0.8	1.3	0.01	3	−1	0.7	42.6	41.1

Heston model via new FT-Q

CPU times for pricing (sec.)

Param.	θ_v	κ_v	σ_v	λ	ρ	v	OR	AL
Case I	0.45	2	0.1	0.5	0	0.2	41.2	39.6
Case II	0.8	1	0.4	1.6	0.7	0.5	45.8	51.2
Case III	0.01	3	0.2	−1	−0.7	0.05	58.1	56.6
Case IV	0.2	0.3	0.75	0	−0.1	0.2	46.8	42.7
Case V	0.3	2.2	0.15	−2	1	0.3	64.0	53.9
Case VI	0.8	1.3	0.01	3	−1	0.7	43.5	44.8

Heston model via FFT

CPU times for pricing (sec.)

Param.	θ_v	κ_v	σ_v	λ	ρ	v	OR	AL
Case I	0.45	2	0.1	0.5	0	0.2	0.92	0.94
Case II	0.8	1	0.4	1.6	0.7	0.5	1.49	0.96
Case III	0.01	3	0.2	−1	−0.7	0.05	1.9	1.88
Case IV	0.2	0.3	0.75	0	−0.1	0.2	1.84	1.53
Case V	0.3	2.2	0.15	−2	1	0.3	3.07	2.9
Case VI	0.8	1.3	0.01	3	−1	0.7	0.9	1.0

¶PRICING PERFORMANCES: Tables for Merton model.

NC = Not computed (due to instability problems).

Merton model via OLD FT-Q

CPU times for pricing

Param.	σ_j	λ_j	μ_j	v	CPU times (sec.)
Case I	0.1	0.5	0	0.2	73.7
Case II	0.4	1.6	0.2	0.5	78.6
Case III	0.2	2	0.5	0.05	NC
Case IV	0.75	0	0.9	0.2	NC
Case V	0.15	0.02	1.5	0.3	62.5
Case VI	0.01	0.9	2	0.7	80.0

Table (continued)

Merton model via NEW FT-Q

CPU times for pricing

Param.	σ_j	λ_j	μ_j	v	CPU times (sec.)
Case I	0.1	0.5	0	0.2	69.3
Case II	0.4	1.6	0.2	0.5	72.2
Case III	0.2	2	0.5	0.05	65.2
Case IV	0.75	0	0.9	0.2	74.4
Case V	0.15	0.02	1.5	0.3	66.1
Case VI	0.01	0.09	2	0.7	68.3

Merton model via FFT

CPU times for pricing

Param.	σ_j	λ_j	μ_j	v	CPU times (sec.)
Case I	0.1	0.5	0	0.2	2.45
Case II	0.4	1.6	0.2	0.5	3.90
Case III	0.2	2	0.5	0.05	3.28
Case IV	0.75	0	0.9	0.2	1.8
Case V	0.15	0.02	1.5	0.3	1.1
Case VI	0.01	0.09	2	0.7	3.56

Not surprisingly, the FFT speed performance, on a set of 4100 prices along the strike, are hardly comparable with the other algorithms performance, as FFT is up to 40 times faster the other quadrature schemes. OLD FT-Q and NEW FT-Q speed differences are negligible. In terms of brute force, the equivalent representations for the formulas does not seem to improve speed. Clearly, the FFT accuracy relies heavily on the number of interpolation points for the discrete Fourier transform and the choice of computing 4100 prices is the best one (following Carr–Madan results) in order to compute accurate prices; this implies that, in an extreme scenario, the computation of a single price in the most accurate way with the FFT algorithm will require exactly the computational time required by thousand of prices, since a computational gain can be obtained only at the cost of a decreasing pricing accuracy. So it is understood that, even if in the speed test the FT-Q schemes are slowed down by the use of iterative loops (although vectorized) in the computations of thousand of prices, they can handle with a comparable speed and in a more accurate way the computations of less prices (for instance, in the order of hundreds), with respect to the lack of accuracy, in terms of interpolation points, shown in these situations by the FFT algorithm.

OLD FT-Q algorithm has serious problems of stability in extended areas of the parameters set of clear financial meaning (for example, long expiry time or high volatility). It is known that the FFT algorithm needs alternative pricing formula to handle values for Strikes far away from the Spot Price level, so an arbitrary choice of the "sensible" level of Strike that governs the choice of the right formula is inherent in the FFT approach and mines the stability of the algorithm. Only the NEW FT-Q algorithm satisfies the stability criterion in a proper way.

Moreover, numerical results show that the NEW FT-Q is the only algorithm that improves considerably its stability performance when an equivalent pricing formula is used. Clearly, to assess algorithms accuracy, one has to choose some "reasonable" values for the recombinant parameters in the FFT approach. In fact, the FFT solution identifies a peculiar choice of the recombinant parameters that cannot be applied tout – court to all the models. Moreover, either the OLD FT-Q or NEW FT-Q, when spanned on the parameters set, give the same prices with a precision of 10^{-3}, so the FFT accuracy has to be valued with respect to the results given by the FT approaches (see also footnote 3). An accuracy test for the FFT algorithm compared with the FT approaches is still under progress, so a definitive answer cannot be given. Keeping in mind this fact, one can notice that the use of a different value (considered as optimal,[4] but not confirmed in other studies) for the damping parameter from the original form of Carr–Madan is needed here in order to maintain a reasonable accuracy. Since this parameter depends heavily from the model structure, a unique choice for a wide class of models like AJD looks somewhat suspicious. Moreover, the damping parameter effect is strictly related with the choice of the discretization grid points,[5] so an optimal assessment is really a tough task far to be solved. Obviously, an accurate calibration of the damping parameter for each model analyzed using the accuracy criterion (for a fixed choice of the discretization grids) will answer to this doubt, but at the present time, this "easy use" of given values casts at least a shadow on the accuracy issue. Eventually, we can consider that this problem, uncomfortable for an operational use, does not exist in the FT approach that can be really used without worries as a black box.

We notice that a recent work (Chourdakis, 2004) enhances the Carr–Madan procedure in terms of the choice of the discretization grid points but the accuracy issue remains actually an open problem.

6.2. Calibration

6.2.1. Algorithms

The standard calibration problem (SSE[6]) for AJD models is performed, i.e.

$$\min_{v,\Phi} \sum_{j=1}^{n} [C_{\text{Market}}(S_t, K_j, \tau) - C_{AJD}(S_t, K_j, \tau, v, \Phi)]^2, \tag{69}$$

where Φ is the set of parameter that characterizes the selected model, S_t, K_j, τ can be spotted on the market, and v is the initial variance level.

An unconstrained non-linear optimization algorithm based on the Reflective Newton Method is used for the calibration, since it is preferred to impose implicit parameters constraints for reasons of numerical stability. Each iteration involves the approximate solution of a large linear system using the method of preconditioned conjugate gradients (PCG).[7]

6.2.2. Empirical Performances

It must be stressed that stability in pricing is crucial to optimize either stability or speed performance in calibration. In fact, the optimization algorithm tries thousand of parameters combinations in its search for the minimum; by doing so even a little instability in a limited area of the parameters set can affect seriously the speed performance since the pricing algorithm is slowed down by numerical instability. This hypothesis is confirmed by the calibration results.

The tests are structured in the following way: we do not use real market prices for the calibration, as an empirical analysis of an option market is beyond the scope of this work. Theoretical prices are used, that are directly computed using the models formulas with the same parameters vectors used in the previous tests, in order to preserve the same initial conditions. Then a model calibration is done from a random starting point (resampled for each run) using the three different approaches. We take only care of the computational performances since the calibration accuracy cannot be a subject of a reliable comparative test without considering the "seed choice" problem, i.e. the correct choice of a starting point and having a complete test of the accuracy of the FFT algorithm.

The following tables show the results:

CALIBRATION PERFORMANCES: Tables for Heston model

With OR we are denoting the use of models original formulas

With EQ we are denoting the use of models equivalent formulas

The calibration target vectors are

Parameters	θ_v	κ_v	σ_v	λ	ρ	v
Run 1	0.8	1.3	0.01	3	−1	0.7
Run 2	0.45	2	0.1	0.5	0	0.2
Run 3	0.01	3	0.2	−1	−0.7	0.05
Run 4	0.2	0.3	0.75	0	−0.1	0.2
Run 5	0.3	2.2	0.15	−2	1	0.3
Run 6	0.8	1.3	0.01	3	−1	0.7

[4] See Schoutens et al. (2004).

[5] See Lee (2004).

[6] Sum of Square Errors.

[7] See Coleman and Li (1994).

Where a* appears, the mean result is not considering some calibration tasks not performed for reasons of numerical instability.

Mean performances

Heston model or			
CPU times for calibration (sec.)			
Prices	OLD FT-Q	New FT-Q	FFT
6	397.8*	55.3	6.35
8	437.2*	52.3	5.97
10	547.0	55.7	6.51
14	581.5	56.7	7.05
16	506.4*	60.1	7.22
18	510.1*	58.1	6.98

HESTON model EQ			
CPU times for calibration (sec.)			
PRICES	OLD FT-Q	NEW FT-Q	FFT
6	382.0*	32.5	6.34
8	459.05	32.8	6.17
10	400.8*	27.5	6.52
14	480.2	30.9	6.99
16	489.2	33.8	6.84
18	436.7*	36.8	6.93

¶CALIBRATION PERFORMANCES: Tables for Merton model

The calibration target vectors are

Parameters	σ_j	λ_j	μ_j	v
Run 1	0.1	0.5	0	0.2
Run 2	0.4	1.6	0.2	0.5
Run 3	0.2	2	0.5	0.05
Run 4	0.75	0	0.9	0.2
Run 5	0.15	0.02	1.5	0.3
Run 6	0.01	0.9	2	0.7

Where a * appears, the mean result is not considering some calibration tasks not performed for reasons of numerical instability

MEAN PERFORMANCES

Merton model via NEW FT-Q			
CPU times for calibration (sec.)			
Prices	OLD FT-Q	NEW FT-Q	FFT
4	623.2*	66.5	9.08
6	605.7	64.2	9.02
8	625.8	68.1	8.89
10	636.7	74.7	9.54
12	658.1	73.3	9.53
14	660.4	77.2	9.67

Clearly, the optimization performance of the FFT approach are not comparable with FT-Q speed. Obviously, one must consider the accuracy problem, since the calibrated parameters that come out from the FFT approach depend heavily from the arbitrary choice of the recombinant parameters, but the implications of this problem have just been considered in the pricing section. The most interesting fact here is the sensible performance gain (up to 10 times) when we switch from a Newton–Cotes to a Gauss–Lobatto quadrature algorithm: since the speed performance of the two algorithms during the pricing can be considered exactly the same, the speed gain in calibration results has to come out from the improved stability of the Gauss–Lobatto scheme.

At this time of the analysis, it must be verified if, by implementing the equivalent pricing formulas that avoid the major numerical flaws, it is possible to considerably improve the calibration performance. This intuition has been proved correct since this approach has performed far better than expected. The table results are clear: the FFT performance are not affected by the use of the alternative pricing formula, or the influence can be considered negligible. The OLD FT-Q scheme seems to benefit in some way of the improved stability, but the interesting fact is the performance boost showed by the NEW FT-Q scheme: in the better conditions, the Gauss–Lobatto scheme applied on the alternative formulas is only three times slower than FFT speed performances.

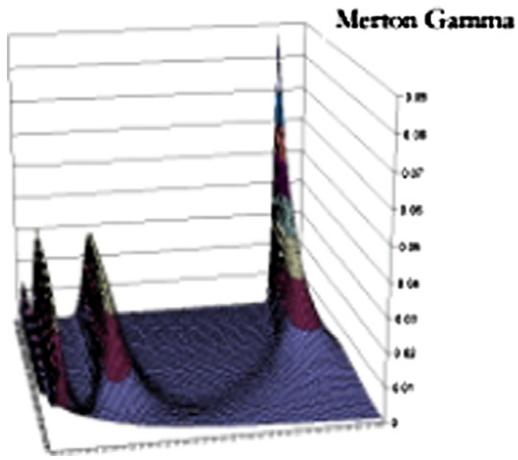
In other words, the calibration times of a classic, revisited, quadrature scheme for the canonical Inverse Levy Formula (that avoids the use of recombinant parameters) are closer than ever to the FFT calibration times.

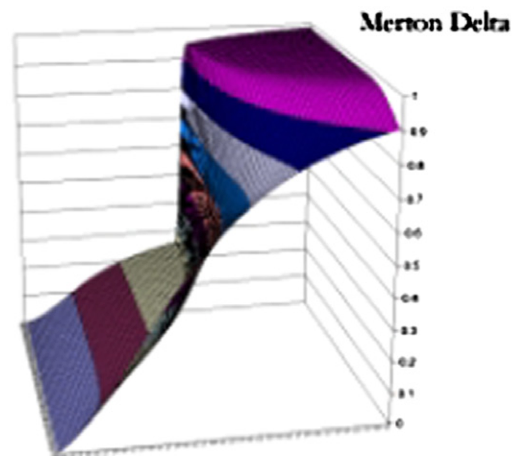
Models extensions: Outlines: The choice of Bakshi–Cao–Chen (Bakshi et al., 1997) model has followed the idea of a natural evolution for Heston model. In fact, this model, without modifying the Heston model structure, allows a stochastic structure of CIR type (i.e. square root) for the free risk interest rate and the presence of jumps in a Merton fashion. Moreover, the BCC model allows a wide generalization at a minimum parameters cost and the linear additive structure of its pricing formula with respect to the stochastic interest rate and jump components, is easy to implement from a numerical point of view.

For BCC model, we have some figures on speed and stability in pricing and hedging:

Pricing speed (sec.)	BCC 97
FFT	1.26
OLD FT-Q	85.4
NEW FT-Q	86.2

Calibration speed (sec.)	BCC 97
FFT	18.32
OLD FT-Q	962.0
NEW FT-Q	93.2





In terms of stability, the BCC pricing formula has not many problems and gives accurate prices when spanned on a vast area of the parameters set; a reasonable explanation is that the increased number of parameters and the presence of two different sources of price volatility gives more flexibility to the formula. An alternative representation for the pricing formula that avoids the computation of bad behaved quantities (as logarithms and fractions) can be easily found and it is shown that this alternative representation improves the stability in pricing and speeds up the calibration procedure. A rigorous battery of tests performed as for Heston and Merton model is still under progress.

6.2.3. Computing Greeks with NEW FT-Q

The presence of closed analytical formula for the Greeks and a stable and reliable approach to compute bad behaved integrands makes possible to study extensively the Greeks Behavior on the parameters set. The NEW FT-Q quadrature scheme is useful to avoid other numerical methods to compute Greeks that are computationally expensive and less accurate and it allows a "quasi real-time" valuation of the parameters non-linear impact on the behavior of the Greeks. In this way, the role of the volatility process parameters and the jumps are perfectly quantified, and their complex relationships with other important parameters (i.e. price of volatility risk, correlation coefficients between processes) that become evident only for extreme (but realistic) values for parameters are fully assessed.

The problem of numerical instability, although very reduced, obviously cannot be ignored. As it appears in the figures below, for extreme parameters value the Greeks show some flaws:

These Greeks (Merton Delta and Merton Gamma) have been computed using the NEW FT-Q scheme applied to the formulas (63), (64). For both cases the parameters used are: $K = 100; \tau = 0.5; r = 0.02; \lambda_j = 0.5, 1.8; \mu_j = 2.5; \sigma_j = 0.1; S_t = [0, 200]; v = [0, 1]$.

The real advantage of this approach is that, even slightly unstable, the NEW FT-Q scheme always "close" the integral, i.e. the charts shows "no hole". In this way. the area of instability can be easily identified and an intensity of instability (related with the anomalous waves showed by the Greeks) can be measured. Then, a simple interpolation can solve the problem to obtain the true pattern of the studied Greek.

7. Conclusions

We revisit a well-known Fourier transform method for pricing and hedging options in affine jump diffusion models, using alternative option pricing formulas for the models and a robust Gauss–Lobatto quadrature scheme replacing the standard Newton–Cotes algorithms previously used in literature. Extensive tests of the new method are performed, in comparison with the classic method and the FFT method, that is a standard tool in numerous empirical studies. The results show an improved stability and speed of computation of our revisited method: the NEW FT-Q scheme is up to 10 times faster the old Fourier transform quadrature scheme and, in the best conditions, its speed is directly comparable to FFT speed performances. Moreover, it must be stressed that our method avoids the greatly underestimated problem of the arbitrary choice of recombinant parameters in the FFT approach and it can be easily used also for the computation of a low number of option prices, a situation that FFT cannot handle without losing in accuracy or in speed; these properties enhance the overall accuracy in pricing and hedging. Eventually, the NEW FT-Q method has allowed a feasible computation and in-depth analysis of the AJD Greeks, making possible a detailed comparative statics study in "quasi" real time. More research has to be done in the assessment of the recombinant parameters in order to improve the accuracy of FFT approach and in the testing of the NEW FT-Q method on more complex models.

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