Probability Performance Scenarios are better: an efficient disclosure of higher moments information from no-arbitrage market implied distributions

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Abstract:

The present work proposes a methodology for the representation of performance scenario in Packaged Retail and Insurance-based Investment Products (PRIIPS), by the means of a no-arbitrage probability table easy to understand for the retail investor. A statistical reconstruction via the method of moments allows to capture the main properties of the PRIIP market implied distribution by identifying the minimum number of descriptive moments needed. A reasonable quantile partition that is effective for representing to the retail investor the complex distributions of structured products characterized by non-linear pay-offs is then proposed.

1. Introduction

As of September 2020, across the European Union (EU) the retail investments industry (comprising structured products, funds and insurance investments) awaits the European Securities Market Association (ESMA) proposed revision to the PRIIPs[1] regime after the consultation process ended in December 2019. These changes will have to be ratified by the European Commission (EC) before a likely implementation date in early 2022. The last revision hit a major stumbling block when the European Securities Authorities (ESAs) could not agree on the draft revisions, as published in July 2020[2].

In the last years, the European Parliament (EP), through Regulation no. 1286 of November 2014, had confirmed that the disclosure requirements concerning the PRIIPS were necessary for the retail investors to understand the risks related to these products, while taking investment decisions. The Regulation was not dedicated to all the financial products, but only to the PRIIPs, and its scope was to ensure the PRIIPs' information disclosure, and consequently, to restore the investors' confidence, damaged by the Great Financial Crisis (GFC)[3] .Indeed, both manufacturers and distributors used to provide for each PRIIP a

[1] Any investment, including instruments issued by special purpose vehicles as defined in point (26) of Article 13 of Directive 2009/138/EC or securitisation special purpose entities as defined in point (a) of Article 4(1) of the Directive 2011/61/EU of the European Parliament and of the Council, where, regardless of the legal form of the investment, the amount repayable to the retail investor is subject to fluctuations because of exposure to reference values or to the performance of one or more assets which are not directly purchased by the retail investor.

[2] ESMA (2020), "Annex to Letter ESA 2020 19-Draft Final Report following consultation on draft regulatory technical standards to amend the PRIIPs KID". Link: https://www.esma.europa.eu/sites/default/files/library/jc_2020_66_priips_rts_draft_final_report.pdf.

[3] Article 4 of Regulation (UE) n. 1286/2014 concerning the Key Information Documents, issued by the European Parliament and Council on the 26th of November 2014.

Link: https://eurlex.europa.eu/legalcontent/EN/TXT/PDF/?uri=CELEX:32014R1286&from=IT.

prospectus. However, this document was considered too long and complex to be completely understood. This discouraged any careful reading.

The EC presented a first version of the Key Information Document (KID) regulatory technical standards (RTS) in 2016[4]. The KID is a standard document which consists of maximum 3 sides of an A4-sized paper, which should be delivered to the retail investors before any purchase. Ideally, through this document, the investors should have access to the key characteristics of each PRIIP, allowing them to make better investment decisions. Indeed, the investors could compare different PRIIP KIDs realized by different manufacturers. Particularly, according to the regulation, the KID is a briefing document, which aims to: provide general information about the product; identify and analyse the level of the risk for each PRIIP, "in the form of a risk class by using a summary risk indicator (SRI) having a numerical scale from 1 to 7"; identify and analyse 4 different payoffs in three different time periods, known as performance scenarios; identify and analyse all the costs related to the PRIIP. It is important to specify that, for the purpose of the risk assessment, the Regulation divide the PRIIPs into 4 categories:

Table 1 – PRIIPs Categories

Category 1	all the high-risk products (i.e. the potential losses are higher than the amount invested), and all those products for which is not easy to compute the level of the risk, because of the lack of historical data or the lack of related benchmarks (i.e. Contract For Difference –CFD)
Category 2	all the products whose payoffs are a linear function of the underlying investments (i.e. mutual funds or ETFs)
Category 3	all the products whose payoffs are not a linear function of the underlying investments (i.e. Structured products)
Category 4	all the products whose values do not depend on factors observable on the market (i.e. Insurance-based products, Guaranteed Interest rate with profit sharing)

In 2017 a new one was introduced to the EP[5], with some amendments concerning the KID document, including a section dedicated to the methodology for assessing and presenting the risk of the PRIIPs. On the 1st January 2018, the Regulation came into effect in all the Member States.

2. The Shortcomings of PRIIPs Performance Scenarios

Previously, several amendments were implemented within the regulation, between 2014 and 2016. Most of them concerned the performance scenarios, as it was (and still is) one of the most critical RTS. Actually, as specified in the regulation, the performance scenarios "*shall be presented in a way that is fair, accurate clear and not misleading"*[6]. Furthermore, the KIDs should provide a forward-looking analysis[7] of the

[4] Commission Delegated Regulation (EU), of 30 June 2016, supplementing the Regulation (EU) n. 1286/2014.

[5] Commission Delegated Regulation (EU), 2017/653 of 8 March 2017, supplementing the Regulation (EU) n. 1286/2014.13

[6] Commission Delegated Regulation (EU), 2017/653 of 8 March 2017, supplementing the Regulation (EU) n. 1286/20.

potential return the investor could get, considering the initial amount invested (usually 10,000 simulations for any currencies, as suggested by the Regulation), over 3 different periods (1 year after the initial investment, half of the recommended holding period, the recommended holding period), under different scenarios. In essence, the scenarios to be implemented are: an unfavourable scenario, a favourable scenario, a moderate scenario. Recently a stress scenario has been introduced to capture all the adverse impacts not included in the unfavourable one. Through the illustration of the potential performances related to a certain investment, the investor could compare them with the ones of other products, and take a more informed investment decision. To compute these performance scenarios, the KIDs producers should follow the guidelines specified within the Regulation. In general, all calculations[8] should be carried out using the historical fund prices, which length depends on the frequency of available data: daily, at least 2 years of available prices; weekly, at least 4 years of available prices; monthly: at least 5 years of available prices.

A risk management measure widely exploited in the RTS in different variations is the Value at Risk (VaR)[9], which is used to compute the maximum potential loss that an investor would expect to incur on a certain investment position. It is a probabilistic measure that captures, with a 97.5% of confidence level, the potential loss exceeding the 2.5% in a specified time horizon N.

Formally:

$$VaR_{0.975} = -12\sigma 2N + 0.025z\sigma\sqrt{N} \quad (1)$$

where σ is the volatility of the logarithmic returns, N is the time horizon for which the VaR is calculated and $0.025z$ equals -1.96.

However, the returns of the investment products are often skewed and their distribution does not follow the Gaussian curve. Since the VaR measure assumes that the returns are normally distributed, then it would lead to inaccurate results while computing the potential risks related to a PRIIP. For this reason, it has been considered the Cornish –Fisher Expansion (CFE[10]) in the methodology for Category 2 PRIIPs. Indeed, this technique is based on the first four moments of the distribution, and can convert a normal variable into a non-normal one.

$$CFE = \left[z_\alpha + \frac{(z_\alpha{}^2-1)}{6}S + \frac{(z_\alpha{}^3-3z_\alpha)}{24}K - \frac{(2z_\alpha{}^3-5z_\alpha)}{36}S^2\right] \quad (2)$$

where S represents the skewness of the product distribution (third moment) and K the excess kurtosis (fourth moment).

Calculations are declined in different ways depending from the category assigned to the PRIIP. Anyway, from a broader perspective, scenarios are the outcome of sophisticated calculations which are based always on the distribution of the historical return data over the past 5 years. With this approach the PRIIPs

[7] See footnote 5, Annex 1: KID Template.

[8] Annex II, see footnote 5.

[9] Duffie, D. and J. Pan, An Overview of Value at Risk. Journal of Derivatives 4 (Spring), 7-49, 1997.

[10] Cornish, E. A.; Fisher, Ronald A. (1938). "Moments and Cumulants in the Specification of Distributions" (PDF). Revue de l'Institut International de Statistique / Review of the International Statistical Institute. 5 (4): 307–320.

KID is implicitly re-introducing past performance an indicator of future outcomes. Early consumer testing[11] specifically designed for the presentation of the scenarios have suggested that a simple graphic incorporating a table or line graph is more helpful than complex designs that involve showing probabilities or a ‘funnel of doubt’. But this battery of consumer testing never attempted to examine whether investors could understand past performance better than scenarios, whether they could draw more meaningful conclusions, or whether they would be more likely to engage with the one instead of the other.

There are other critical weakness in the technical details of the hybrid approach based both on forward simulations and historical data currently in discussion. The removal of discrete calendar year past performance figures from the KID would likely take away the visibility of the simple fact that a product may make gains in some years and losses in others – a point that the KID consumer test had indicated investors would look at. Moreover, since the scenarios are based on the performance during the previous 5 years, they will be directly related to the market experience over that period. If that captured strongly rising markets then even the unfavorable scenario could appear optimistic. In contrast, after a more challenging period, the scenarios would suggest that investors should expect continued poor returns. In reality, this is the opposite of what investors should expect. Periods of strongly rising markets are more likely to precede periods of weaker returns, and vice versa. This is also true when only past performance is shown but past performance is factual and does not offer a forecast that is the opposite of what is most likely to happen.

In light of a barrage of negative reactions based mainly (but not only) on these shortcomings, it has been decided to delay extending the PRIIP disclosure until the end of 2021 and take the extra time to fix the issues and address all concerns[12].

3. An implied probability approach to performance scenarios

3.1 The theoretical rationale

The price of a contingent security, being a variable that is dependent on the price of the underlying asset at a future date, contains information connected with the probability estimates made by market operators about the dynamics of the underlying. Black & Scholes (1973) have demonstrated in a seminal work that derivative contracts like options can be generally priced under the no-arbitrage hypothesis in a way that is independent from the investors preferences. In that context, the price of an European *call* option can be calculated as the discounted expected value of the option values at maturity with respect to the *no-arbitrage* probability measure, i.e. the measure that guarantees the absence of arbitrage between quoted prices. Formally:

$$C_T = e^{-rT} E^{\mathrm{P}}\left[max\left(S_T - K, 0\right)\right] = e^{-rT}\int_0^{\infty} max\left(S_T - K, 0\right) f\left(S_T\right) dS_T \qquad (3)$$

[11] European Commission (2014), “Consumer testing study of the possible new format and content for retail disclosures of packaged retail and insurance-based investment products”, Final Report by London Economics and IPSOS, European Union. MARKT/2014/060/G for the implementation of the Framework Contract n° EAHC-2011-CP-01. Link: https://ec.europa.eu/info/sites/info/files/consumer-testing-study-2015_en.pdf.

[12] There are still questions whether it will require a change in the overarching regulatory framework itself (at Level 1) or if issues can be addressed through changes in the technical implementation of that regulatory framework (Level 2). Legislators will naturally steer towards the latter as it poses fewer challenges but some of the fixes may well need the former.

where C_T is the option price, S_T is the underlying price at time *T,* maturity date of the option, *K* is the Strike price, *r* is the *risk-free* interest rate (current account or short term interest rate like the *Overnight Indexed Swap*, OIS) supposed as constant, and $E^P[max(S_T - K, 0)]$ denotes the expected value with respect to the *no-arbitrage* probability measure *P*.

Conditionally to the knowledge of $f\left(S_T\right)$, the probability density function of the underlying, equation (3) can be used to price the *call* option. By an inverted argument, in Breeden e Litzemberger (1978), it is shown that it is possible to derive an explicit representation for $f\left(S_T\right)$ multiplied by a discount factor, by differentiating equation (1.1) two times with respect to the Strike price, i.e.:

$$\frac{\partial C_T}{\partial K^2} = e^{-rT} f\left(S_T\right) \quad (4)$$

The knowledge of a sufficient number of market prices C_T for different Strike prices *K* allows to obtain the implicit probability function $f\left(S_T\right)$.

Alternatively, one of the standard methodologies to use formula (4) (implemented in this work) requires that the underlying asset S_T follows a particular parametric stochastic process. This implies by definition a partial preliminary knowledge of the functional form assumed by the probability density function at maturity. Conditionally to the knowledge of the parameters that completely characterize the stochastic process for S_T, it is possible in certain circumstances to derive an explicit formula for the option price by exploiting equation (1.1). The parameters are then determined explicitly by minimizing a specific distance between the observed option prices and the theoretical prices given by the model (calibration procedure).

The major risk that underlies the choice of a specific parametric process for the dynamics of the underlying is that the probability distribution implied by the model can be not enough flexible to capture all the statistical features of the “empiric” distribution of the financial product.

To avoid this problem, it is important to select with accuracy the stochastic model more able to describe the dynamics of the underlying, on the basis of the statistical patterns and empirical regularities shown by the financial asset during its life, of its duration and financial engineering.

The information embedded in the implied probability distribution has the indisputable advantage of being always updated and reactive to the variable market conditions, having considered its direct connection with the quoted prices of liquid assets. When the market volatility is high, this technical properties is of fundamental importance in the perspective of the investor protection, especially with respect to metrics currently in evaluation for PRIIPS that are based on historical data or economic and accounting information, e.g. the *rating* estimated for the evaluation of the credit risk of the issuers of financial products.

It’s not the first time that the no-arbitrage approach used to build market implied probability distributions is exploited in the representation PRIIPs performance scenarios. In one of the last iterations of the Regulation[13] the market risk-assessment of category 3 PRIIPS was explicitly calculated by using no-arbitrage expectations.

[13] COMMISSION DELEGATED REGULATION (EU) 2017/653 of 8 March 2017 supplementing Regulation (EU) No 1286/2014 of the European Parliament and of the Council on key information documents for packaged retail and insurance-based investment

The proposal described in this paper for the PRIIPs performance scenario exploits extensively some results of previous works of Minenna et al. (2009) and Minenna (2011) that are focused on the definition and implementation of a quantitative approach to the risks' transparency of non-equity products (so called "three pillar" risk-based approach). This methodology provides an informative set comprehensible, concise and effective to support retail investors in taking their investment decisions. In this perspective, the risk-based approach sets out an objective methodology to determine and represent three synthetic risk indicators (so cited three pillars) — all calculated using probabilistic tools — which meet in a clear, meaningful and internally consistent way to the information needs that emerge when one is interested in comparing and choosing among the various non-equity products:

• the price unbundling and the probabilistic scenarios (so-called first pillar[14]);

• the degree of ongoing risk (so-called second pillar[15]);

• the recommended time horizon of investment (so-called third pillar[16]).

The methodology presented here fits perfectly the problem of representing the performance scenarios for category 3,4 PRIIPS, with a potential extension to category 1 PRIIPS.

3.2 The implied probability distribution quantiles as significant thresholds for risks transparency

The PRIIPs performance scenarios actually in discussion can be thought as a single point sampled from the probability distribution itself, properly rescaled. Therefore, it's straightforward to infer that, when the

products (PRIIPs) by laying down regulatory technical standards with regard to the presentation, content, review and revision of key information documents and the conditions for fulfilling the requirement to provide such documents.

Link: https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32017R0653&from=EN.

[14] The first pillar of the risk-based approach relies on two complementary tables, respectively the financial investment table and the table of the probabilistic scenarios, to extract from the risk-neutral density of the non-equity product the core information about its value considered at two specific points in time: the issue date and the end of the recommended investment time horizon.

[15] The second pillar of the risk-based approach is the degree of risk. Unlike the first pillar which looks at two specific points in time, this synthetic indicator summarizes the overall riskiness of the product throughout the full period spanned by its recommended time horizon. To this end, by working on the simulated trajectories of the product's value process used by the first pillar it is possible to analyze their variability through a meaningful and straightforward risk metric: the volatility. The degree of risk is obtained by comparing this risk metric against an optimal grid of increasing volatility intervals, and this information is then conveyed to investors by mapping the volatility figure into an ordered qualitative scale of risk classes endowed with a high signalling power.

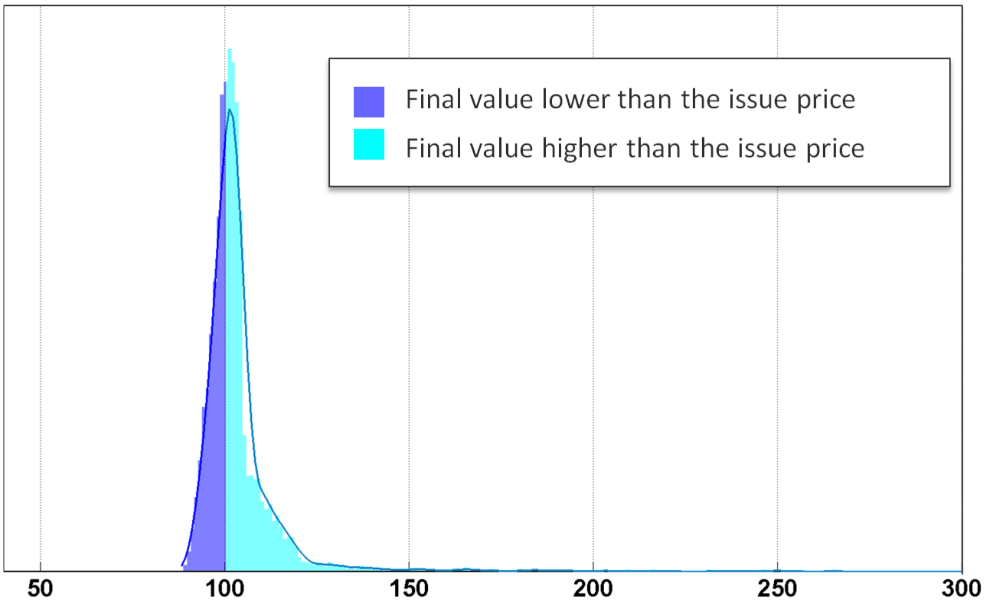
[16] The third pillar is the recommended investment time horizon. This indicator expresses a recommendation on the holding period of the non-equity product, formulated in relation to its specific financial structure and regime of costs. For all non-equity products that instead do not fit this frame, the recommended investment time horizon is determined according to the exogenous criterion of the costs recovery given their riskiness. This criterion is somewhat similar to put at zero the target return of the investor and, in this perspective, the recommended investment time horizon indicates the minimum period within which the costs incurred may be reasonably amortized, taking into account the risks of the product. By applying the theory of the first passage times of a stochastic process for a given barrier and using the same trajectories of the product's value process behind the first and the second pillars it is possible to determine the cumulative probability function of the first times when the value of the product hits a barrier corresponding to the event of costs recovery. This methodology returns a robust indicator, that gives longer horizons for products with increasing risks.

number of sampled points increase, the quantity of information about the distribution's structure tends naturally to grow. By following this line of reasoning and by using the proper choice of a certain number of reference thresholds, it would be possible to identify some key events that could be relevant for the investor from a financial point of view and at the same time could be easily understood.

The number of the events to be identified should allow an effective reading of the principal statistical features of the distribution (e.g. multimodality, asymmetry) but they would need to be limited at most to 3,4 sub-partitions of the distribution, in order to facilitate at the maximum grade the comprehension of the information.

Specifically, the event connected with the recovery of the invested capital is of great relevance for the investor and embeds in a natural way the information related to the exposure to the issuer credit risk; moreover it describes correctly the chance of failure of specific mechanisms of guarantee or protection. Another advantage is that the threshold identification is unique and founded on absolute references (see Figure 1).

Figure 1 – Partition of the risk-neutral probability density of the financial product with respect to the point of zero return



A more refined partition should consider also the comparison of the product's performance with that of the riskless asset in order to take care of the financial value of the time. In this perspective, a possible solution, that has been described and implemented in Minenna et al. (2009), could be to select some thresholds of strong financial meaning connected with the benchmark performances of the risk-free asset; this partition would identify the probability that the product has to beat the benchmark performances, or to be in line, or to lose with respect to the risk-free asset.

In this case, it's obvious that the identification of the thresholds is somewhat discretionary but always related to the possible variation range for the risk-free asset (see Figure 2).

It's immediate to appreciate that the partition of the original distribution in the described 4 macro-events and its dynamic update allow to easily incorporate the information related to the term structure and volatility of the interest rates.

Moreover, the ability of the specified partition to capture the distributional features of the product can be strongly improved by representing, together with the probability, an absolute indicator of performance, e.g. the mean or the median of the reference subset (see for an example Table 8). In this way, the investor is able to perceive also the quantitative impact of the risks on the invested capital, in order to have a more complete information about the risks involved in the investment.

Figure 2 – Partition of the risk-neutral probability density of the financial product with respect to the point of zero return and two fixed positive thresholds α_1 *and* α_2

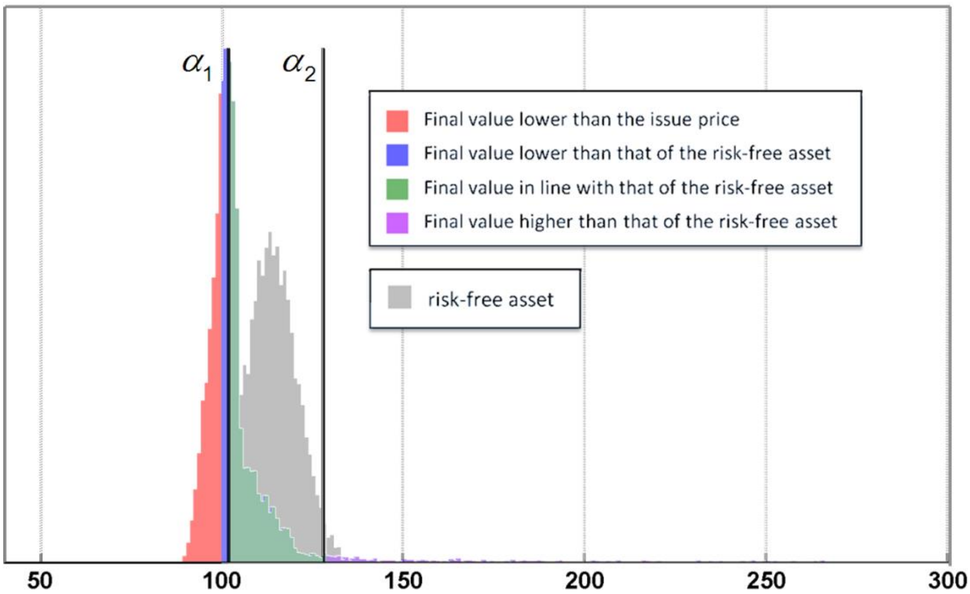


Table 2 – Mean values of performance for the identified partition

$E^{P}\left(S_T \mid S_T < 100\right) =$	$\dfrac{1}{P\left(S_T < 100\right)} \int_{-\infty}^{100} x f_{S_T}(x)\, dx$
$E^{P}\left(S_T \mid 100 \leq S_T < \alpha_1\right) =$	$\dfrac{1}{P\left(100 \leq S_T < \alpha_1\right)} \int_{100}^{\alpha_1} x f_{S_T}(x)\, dx$
$E^{P}\left(S_T \mid \alpha_1 \leq S_T < \alpha_2\right) =$	$\dfrac{1}{P\left(\alpha_1 \leq S_T < \alpha_2\right)} \int_{\alpha_1}^{\alpha_2} x f_{S_T}(x)\, dx$
$E^{P}\left(S_T \mid S_T \geq \alpha_2\right) =$	$\dfrac{1}{P\left(S_T \geq \alpha_2\right)} \int_{\alpha_2}^{\infty} x f_{S_T}(x)\, dx$

The overall set of additional indicators can be represented by means of a table. In the following, the described methodology is implemented for 4 different theoretical financial products.

3.3 Four different financial products and the information content of the first moment

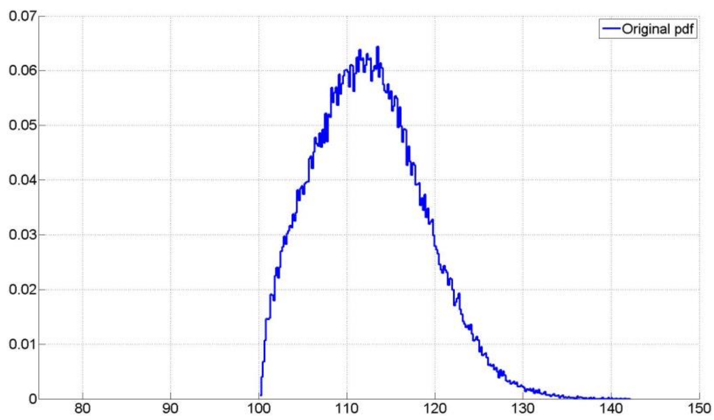
Usually we are accustomed to think at financial product through labels: bonds, equities, swaps are simply names that often characterize the same type of pattern in terms of probability density, i.e. the complete description of the potential returns of the product associated with a specific probability of occurrence. In the following we try to identify products that are substantially different, having payoffs that generate returns probability distribution deeply distinctive.

In this perspective, the choice of theoretical products is the most natural one, since the complete control of the design – from the pay-offs to the discount factors up to the models for the underlying – allow us to exacerbate the pros and cons of the proposed methodology. The application of this probabilistic approach to real-world's products is a forthcoming issue of research but at the present is out of the scope of this paper.

The use of no-arbitrage probability tables is a risk-management techniques that can be exploited to highlight a product's key characteristics in terms of risk. Under no circumstances it has to be intended as a tool of prediction for the future performance of the product. No-arbitrage probabilities represents distorted expectations due to the reasonable risk aversion of the retail investor; this is a well-established result in the literature (see section 5 for a comprehensive discussion). As a consequence, realized performances of real-world products are not so important in testing and evaluating probabilistic no-arbitrage scenarios, since no reasonable conclusion about the methodology of risks disclosure can be derived by their observation.

In the Table 3 below, a synthetic description of each financial product considered is given, together with the plot of the associated probability densities[17].

Table 3 – Description and market implied probability density function of 4 different financial products

Low risk floater coupon bond Maturity: 5 Year Fair Price: 100 Coupon: Floater, paid on an annual basis, connected to a common interest rate Risk Factors: Interest Rate volatility Probability of Default of the Issuer: negligible	

[17] A more detailed description of the products useful for replication purposes is presented in the Annex.

Defaultable fixed coupon bond

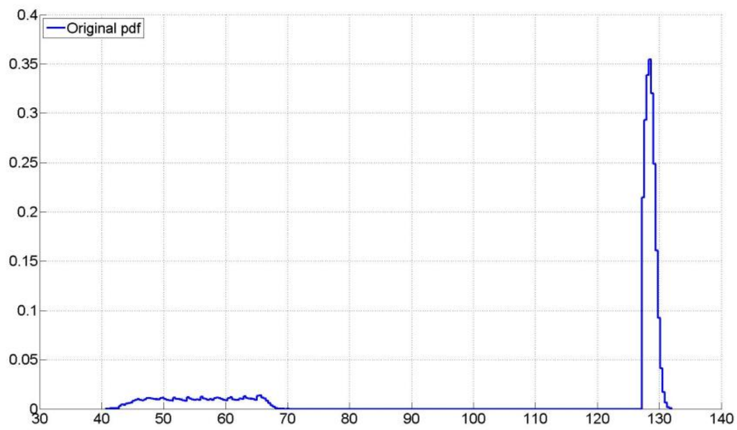
Maturity: 5 Year

Fair Price: 100

Coupon: 4,5% paid on an annual basis

Risk Factors: Interest Rate volatility, Default Risk of the Issuer

Probability of Default of the Issuer: significant



Variable Protection Portfolio Insurance

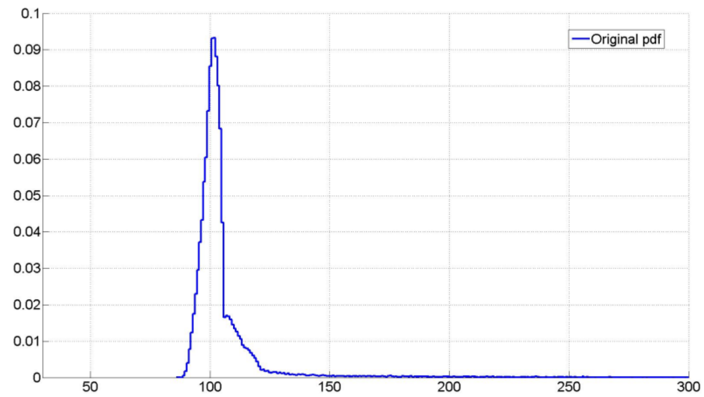
Maturity: 5 Year

Fair Price: 100

Risk Factors: Interest Rate volatility, Market Risk

Narrative Description of the Product:

VPPI technique is aimed at protecting the initial value of the financial investment over a specified time horizon and obtaining possible gains by limited exposure to the equity markets.



Index Linked Certificate

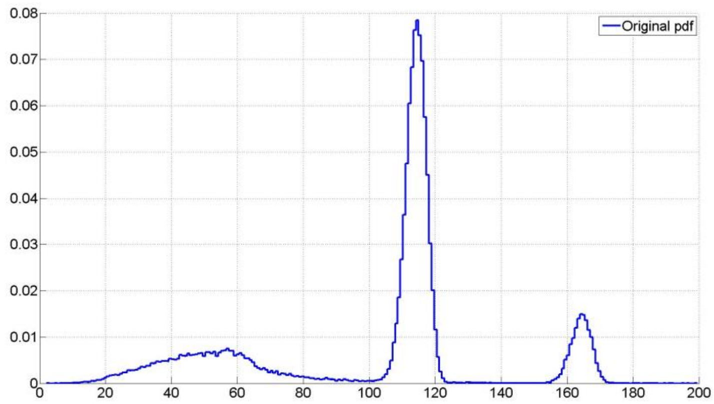
Maturity: 5 Year

Fair Price: 100

Risk Factors: Interest Rate volatility, Market Risk

Narrative Description of the Product:

The index-linked certificate is characterised by a complex financial engineering that makes intensive use of different derivatives components. These derivatives link the performances of the product to the variability of an equity index.



As can be appreciated at a first glance, the 4 products are characterized by the same fair price that therefore does not offer any help to the investor's selection process. Different ways to increase the information available to the investor are found in financial literature: for example, as reported in Minenna (2011), it's possible to exploit the portfolio replication principle to decompose the information provided by the fair price without considering any additional moment. In this way the two key contributors to the fair value, i.e. the risk-free component and the risky one are highlighted and represented by using an uniform criterion at no further computational cost.

The formal passages to implement such decomposition are reported in the following. For the proofs and further details, please refer to Minenna (2011).

Let S_t be any financial product; then it can be replicated by a portfolio composed of the associated risk-free security X_t and of a zero-value swap $swap_t$ which transforms the cash flow structure of the risk-free security into the cash flow structure of the product itself, i.e.:

$$S_t = X_t + swap_t \qquad (5)$$

where

$$swap_0 = 0 \qquad (6)$$

In the context of the split between the risk-free and the risky component, the two legs that characterize the swap are appropriately restructured to uniquely qualify the risky component of the financial product. Intuitively, this restructuring of the two legs is functional to ensure that their values at maturity represent the cases where the financial product outperforms the corresponding risk-free security and where it performs worse that the security, respectively.

Formally, in order to make the breakdown of the fair value, the random variable $swap_T$ is decomposed as follows:

$$swap_T = swap_T^+ - swap_T^- \qquad (7)$$

where

$$swap_T^+ = \begin{cases} swap_T & \text{if } swap_T > 0 \\ 0 & \text{if } swap_T \le 0 \end{cases} \qquad (8)$$

and

$$swap_T^- = \begin{cases} 0 & \text{if } swap_T > 0 \\ -swap_T & \text{if } swap_T \le 0 \end{cases} \qquad (9)$$

Moreover, the following equality holds:

$$E^{\mathrm{P}}\left(swap_T^+\right) = E^{\mathrm{P}}\left(swap_T^-\right) \qquad (10)$$

From the decomposition (7) it follows that the discounted expected value of any of these two components is an estimate of the value of the risky component of S_0. In formal terms:

$$S_0^{risky} = e^{-rT} E^{\mathrm{P}}\left(swap_T^+\right) = e^{-rT} E^{\mathrm{P}}\left(swap_T^-\right) \qquad (11)$$

and accordingly:

$$S_0^{rf} = S_0 - S_0^{risky} \quad (12)$$

If we try to apply this straightforward procedure to the described 4 financial product, we obtain the results reported in the Table 4 below:

Table 4 – Decomposition of the fair value of 4 different financial products

<table>
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Low risk floater coupon bond

Maturity: 5 Year

Fair Price: 100

Coupon: Floater, paid on an annual basis , connected to a common interest rate

Risk Factors: Interest Rate volatility

Probability of Default of the Issuer: negligible

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<td>
<table>
<tr><td>Risk-Free Component (A)</td><td>100</td></tr>
<tr><td>Risky Component (B)</td><td>0</td></tr>
<tr><td>Fair Value (A+B)</td><td>100</td></tr>
</table>
</td>
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Defaultable fixed coupon bond

Maturity: 5 Year

Fair Price: 100

Coupon: 4.5% paid on an annual basis

Risk Factors: Interest Rate volatility, Default Risk of the Issuer

Probability of Default of the Issuer: significant

</td>
<td>
<table>
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<tr><td>Risky Component (B)</td><td>11.98</td></tr>
<tr><td>Fair Value (A+B)</td><td>100</td></tr>
</table>
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<td>

Variable Protection Portfolio Insurance

Maturity: 5 Year

Fair Price: 100

Risk Factors: Interest Rate volatility, Market Risk

Narrative Description of the Product:

VPPI technique is aimed at protecting the initial value of the financial investment over a specified time horizon and obtaining possible gains by limited exposure to the equity markets.

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<table>
<tr><td>Risk-Free Component (A)</td><td>94.99</td></tr>
<tr><td>Risky Component (B)</td><td>5.01</td></tr>
<tr><td>Fair Value (A+B)</td><td>100</td></tr>
</table>
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Index Linked Certificate

Maturity: 5 Year

Fair Price: 100

Risk Factors: Interest Rate volatility, Market Risk

Narrative Description of the Product:

The index-linked certificate is characterised by a complex financial engineering that

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<table>
<tr><td>Risk-Free Component (A)</td><td>90.10</td></tr>
<tr><td>Risky Component (B)</td><td>9.90</td></tr>
<tr><td>Fair Value (A+B)</td><td>100</td></tr>
</table>
</td>
</tr>
</table>

makes intensive use of different derivatives components. These derivatives link the performances of the product to the variability of an equity index.	

3.4 The problem to extract information from higher moments of implied probability distributions

It's clear that we need to take some steps beyond if we want to enrich the informative set provided to investors with an increased detail. However, as long as figures displayed to investors in that table are only valid "on average" due to the fact that they are based only on the information content of the first moment of the probability distribution, it still remains a huge information gap, which requires to develop further indicators that, starting from the no-arbitrage density, provide a clear and objective illustration of the levels of the possible performances and of their variability.

In this perspective, the information content of moments higher than one allow for the appreciation of the degree of randomness characterising the performances of a given product.

As highlighted in equation (3), the price is a measure strictly correlated with the mean or the first moment of the probability distribution, without considering for a while the impact given by the stochastic discount factors. Also the expected internal rate of return (IRR), defined as the mean of the possible several internal rates of return of the financial investment is a measure obtained by the first moment of the probability distribution.

It's well understood that the first moment cannot convey any useful information concerning the risk of a security, a part from the trivial case of a riskless asset. But also the second moment, in general, could be an insufficient statistics.

The more the original probability distribution is characterized by complex and irregular patterns and peculiar statistical features like multimodality, asymmetry or kurtosis, the less we can confine our information set to the first two moments of the probability distribution.

To demonstrate this thesis, a reconstruction analysis of the original probability distributions related to a products sample has been performed by exploiting the information contained in a limited number of moments. The used methodology is based on well known convergence results about the solution of the classical problem of moments for a probability distribution (Shohat et al. 1943. Totik, 2000), rearranged in an approximated form in a context where only a finite number of moments is available (Gavriliadis, Athanassoulis 2009).

<u>3.4.1 A Reconstruction Analysis from a finite set of moments</u>

In this section we follow strictly (Gavriliadis, Athanassoulis 2009). For more details, please refer to their paper. Let $\left(\mu_0, \mu_1, ..., \mu_{2k}\right)$ be the known 2k moments of the probability distribution of a financial product, where:

$$\mu_k = \int_{-\infty}^{\infty} x^k dF\left(x\right) \qquad (13)$$

It is possible to identify precise tail delimiters x_l, x_r such that:

$$x_l : F(x_l) < \varepsilon_l$$
$$x_r : F(x_r) < 1 - \varepsilon_r \tag{14}$$

where $\varepsilon_l, \varepsilon_r$ are very small positive quantities.

In order to derive these left and right tail delimiters, some intermediate results are needed, that exploit the properties of orthonormal polynomials (see Szego, G. (1959)). In fact, a sequence of orthonormal polynomials can always be defined in terms of the *2k* moments of the original probability distribution, i.e.:

$$P_k(x) = \frac{1}{\sqrt{H_{2k}H_{2k-2}}} D_k(x) \tag{15}$$

where

$$D_k(x) = \det\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{k-1} & \mu_k & \cdots & \mu_{2k-1} \\ 1 & x & \dots & x^k \end{bmatrix} \tag{16}$$

and

$$H_{2k} = \begin{vmatrix} \mu_0 & \cdots & \mu_k \\ \cdots & & \\ \mu_k & \cdots & \mu_{2k} \end{vmatrix} \tag{17}$$

is an Hankel matrix, with $H_{-2} = H_0 = 1$.

Having built the above sequence of orthonormal polynomials $P_k(x)$, the Christoffel function $\lambda_k(x)$ is then defined as:

$$\lambda_k = \frac{1}{\sum_{n=0}^{k} \left|P_n(x)\right|^2} \tag{18}$$

The Christoffel function, evaluated at points set equal to the roots of the orthonormal polynomials $P_k(x)$, is then used to state the following powerful result, that provide valuable information about the probability mass between two any distinct roots $x_{k,l}, x_{k,m}$ $1 \le l < m \le k$ (see Shohat J. A., Tamarkin J. D. (1943)):

$$\sum_{i=l+1}^{m-1} \lambda_k(x_{k,i}) \le \int_{x_{k,l}}^{x_{k,m}} dF(x) \le \sum_{i=l}^{m} \lambda_k(x_{k,i}) \tag{19}$$

From equation (19), Gavriliadis and Athanassoulis (2009) proved that, having defined the quantities:

$$L_{k,i}(F) = L_{k,i}\left(\{\mu_n\}_{n=1}^{2k}\right) = 1 - \sum_{j=i}^{k} \lambda_k\left(x_{k,j}\right) \tag{20}$$

$$U_{k,i}(F) = U_{k,i}\left(\{\mu_n\}_{n=1}^{2k}\right) = \sum_{j=1}^{i} \lambda_k\left(x_{k,j}\right) \tag{21}$$

then the following set of inequalities holds:

$$L_{k,i}(F) \le F\left(x_{k,i}\right) \le U_{k,i}(F) \tag{22}$$

Relationship (22) clearly identifies the bounds of the cumulative distribution function for a hypothetical financial product exploiting only the information contained in the first $2k$ moments (see Figure 3, where $k = 6$).

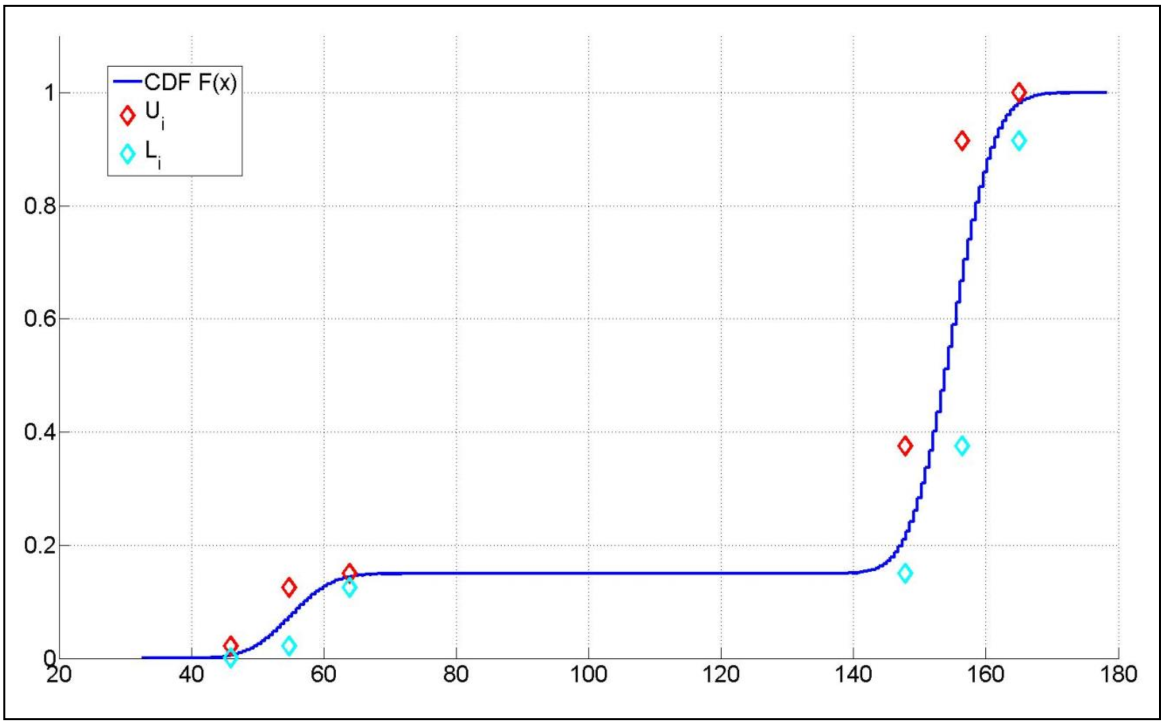
From equation (22) numerous useful estimates can be immediately derived for the main probability mass and the tails (Gavriliadis, Athanassoulis (2009)).

Between the identified bounds, more information about the probability density can be extracted from the available moments. Following Totik (2000), it is possible to characterize the asymptotic behaviour of the Christoffel function:

$$\lim_{k\to\infty} k\lambda_k(x) = \pi\sqrt{(x-a)(b-x)}\, f(x) \tag{23}$$

where $f(x)$ is a positive bounded function on the real line between the interval $I = (a,b)$.

Figure 3 - Original Cumulative Distribution Function and CDF Upper and Lower Bounds



For *k* large enough, when $f(x)$ is a probability density function, equation (23) can be approximated by the following relationship:

$$f(x) \approx k\lambda_k(x)\left(c_0\pi\sqrt{(x-a)(b-x)}\right)^{-1} \tag{24}$$

when $x \in (a,b)$ and where c_0 is a normalizing factor. The role of c_0 is clear when the subsequent corollary of (24) is easily obtained:

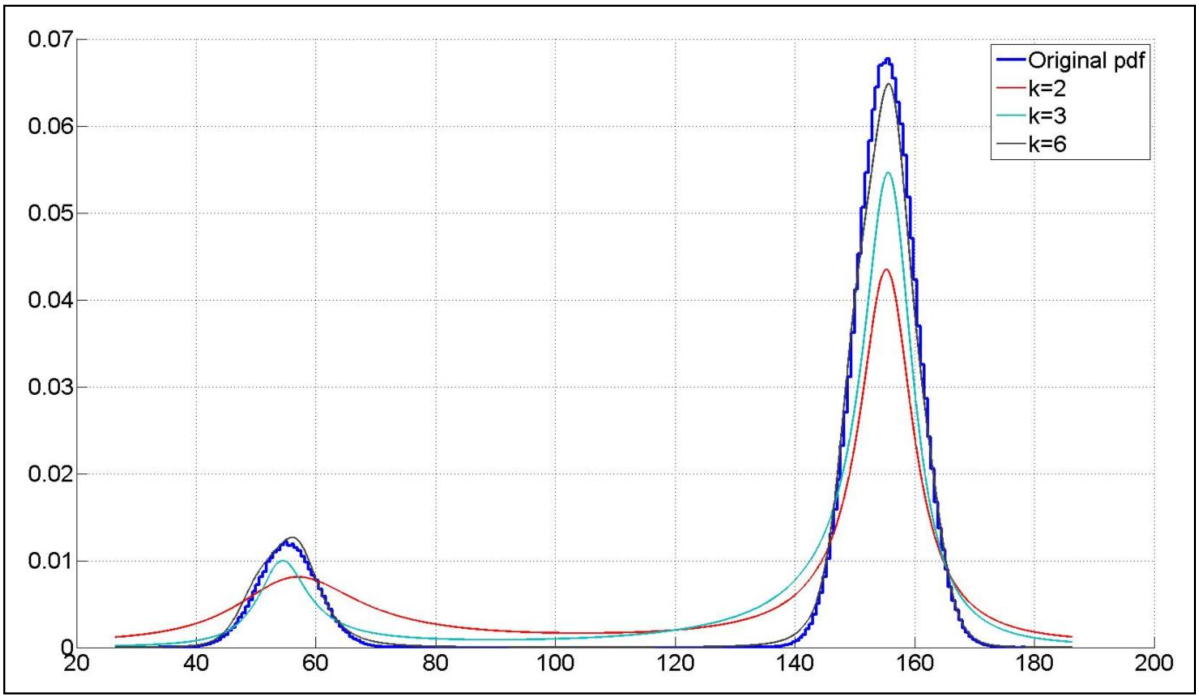
$$F(x) \approx \frac{k}{c_0\pi}\int_a^x \left((u-a)(b-u)\right)^{-\frac{1}{2}} \lambda_k(u)\,du \tag{25}$$

In this case $F(x)$ is the cumulative distribution function and equation (25) suggests us a straightforward way to approximate it. c_0 is then properly chosen to allow (25) to be very nearly 1 when $x = b$.

Figure 4 shows the original probability density and their reconstructed versions for $k = 2,3,6$.

In the next paragraph we will test extensively the accuracy of this method for the 4 financial products that by construction should originate probability densities with very different shapes.

Figure 4 - Original probability density function and 2k-moments reconstructions



4. Results

<u>4.1 Moments Reconstruction</u>

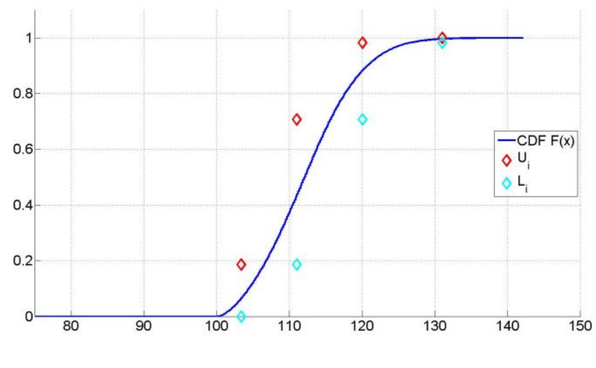
The reconstruction methodology described in section 3.4.1 should be able to quantify the information content that can be extracted from a given set of moments of various probability density patterns: e.g. unimodal and symmetric, asymmetric with extreme tails, multimodal and high kurtosis distributions and so on, in order to properly design supplementary quantitative indicators.

Tables 4, 5, 6 and 7 synthesise the results of the reconstructability tests performed on the products described in Table 3. Each table reports:

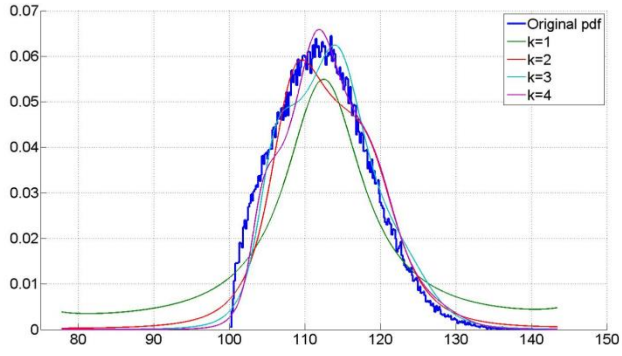
- in the upper left: the CDF chart comprehensive of the localization of the main mass and tails intervals between the tail delimiters $L_{k,i}(F), U_{k,i}(F)$;
- in the upper right: the original probability density of the product superimposed with the reconstructed one by using an increasing number of moments;
- in the mid-section: some relevant statistics for a chosen value of the parameter k, i.e. the roots $x_{k,i}$ of the orthonormal polynomial $P_k(x)$, the numerical values for $L_{k,i}(F), U_{k,i}(F)$ and the value of the CDF in $x_{k,i}$;
- in the lower section: some comments on the ability shown by the methodology to capture the most relevant features of the probability density $U_{k,i}(F)$;

Table 4 – Reconstruction Analysis for the low risk floater bond

Original Cumulative Distribution Function and CDF Upper and Lower Bounds



Original probability density function and 2k-moments reconstructions

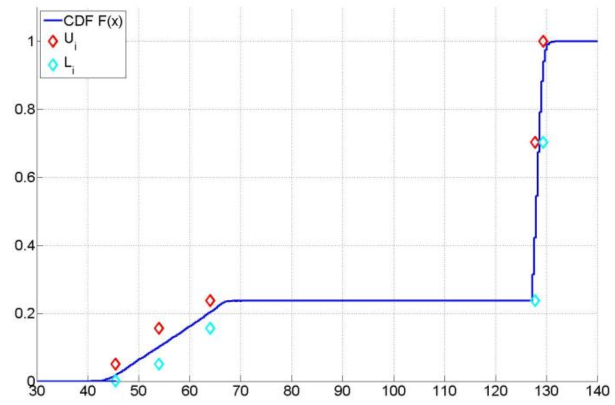


Degree of $P_k(x)$	Roots of $P_k(x)$	Lower bound $L_{k,i}(F)$	CDF $F(x)$	Upper bound $U_{k,i}(F)$
$k=4$	$x_{4,1}=103.387240805$	2.220446049e-16	0.0668984948	0.186385493
	$x_{4,2}=111.048060221$	0.186385493	0.438313345	0.706336578
	$x_{4,3}=120.082610541$	0.706336578	0.885365046	0.982381467
	$x_{4,4}=131.042928137$	0.982381467	0.996471726	1

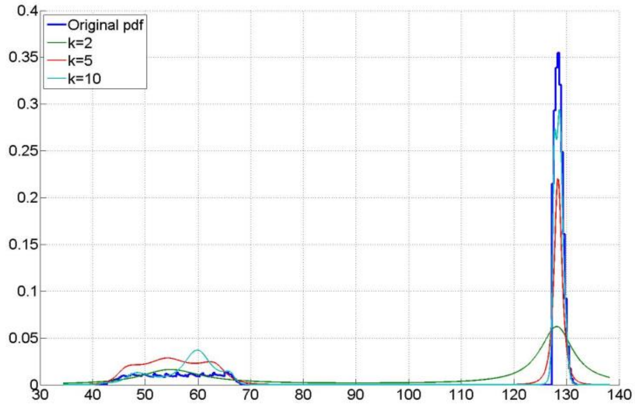
The PDF is characterized by an almost symmetric, smooth behaviour. With only $2k = 2 \cdot 1 = 2$ moments the methodology is able to reproduce the density pattern, with a limited error concentrated mainly in the left tail. Higher moments reconstructions are able to capture the shape of the left tail at the cost of losing accuracy in the modelling of the main probability mass.

Table 5 – Reconstruction Analysis for the defaultable fixed coupon bond

Original Cumulative Distribution Function and CDF Upper and Lower Bounds



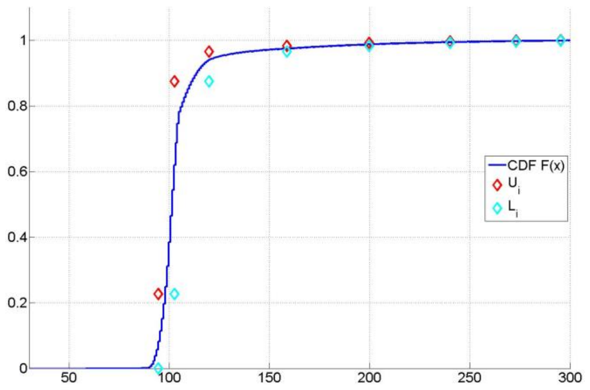
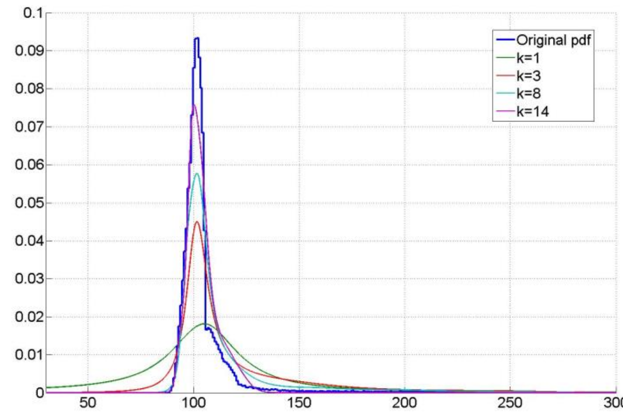
Original probability density function and 2k-moments reconstructions



Degree of $P_k(x)$	Roots of $P_k(x)$	Lower bound $L_{k,i}(F)$	CDF $F(x)$	Upper bound $U_{k,i}(F)$
$k=5$	$x_{5,1} = 45.4059514471$	5.332656538e-11	0.018563610	0.050013330
	$x_{5,2} = 53.968422337$	0.0500133309	0.102785466	0.155301740
	$x_{5,3} = 63.974355092$	0.155301740	0.204211223	0.236431996
	$x_{5,4} = 127.751407552$	0.236431996	0.484319135	0.702725109
	$x_{5,5} = 129.355613138$	0.702725109	0.937680931	0.999999999

The information contained in 4 moments is enough to recognize the presence of two distinct modes, the first very large and the second more narrow. The distance between the two modes is correctly represented also with the low order reconstruction. More moments (up to 20) are needed to properly model the amplitude of the main mode (the second one).

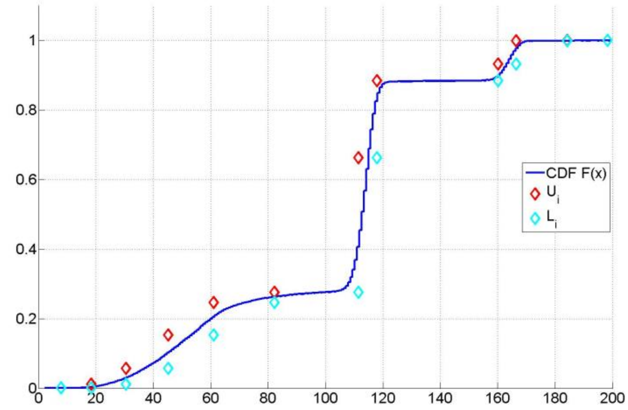
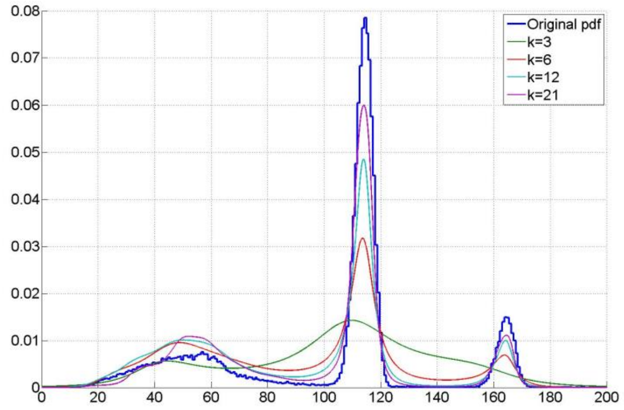
Table 6 – Reconstruction Analysis for the VPPI (Variable Protection Portfolio Insurance)

Original Cumulative Distribution Function and CDF Upper and Lower Bounds	*Original probability density function and 2k-moments reconstructions*
	

Degree of $P_k\left(x\right)$	Roots of $P_k\left(x\right)$	Lower bound $L_{k,i}\left(F\right)$	CDF $F\left(x\right)$	Upper bound $U_{k,i}\left(F\right)$
$k=8$	$x_{8,1}=$ 94.4117485755	2.252072306e-09	0.077545917	0.226620424
	$x_{8,2}=$ 102.470887138	0.226620421875	0.636867711	0.874652785
	$x_{8,3}=$ 119.921155289	0.874652783059	0.942241934	0.966323663
	$x_{8,4}=$ 158.630665061	0.966323661258	0.975519716	0.982812554
	$x_{8,5}=$ 199.759062823	0.982812552589	0.988058261	0.992361721
	$x_{8,6}=$ 240.150491941	0.992361718990	0.995045639	0.997032695
	$x_{8,7}=$ 273.063871337	0.997032693063	0.998353555	0.999235713
	$x_{8,8}=$ 295.350513022	0.999235711218	0.999699631	1

The original distribution is markedly skewed, with a long right tail. The information contained in the first two moments allow to distinguish only a slight asymmetry, but it is not enough to reproduce the thin shape of the left tail and to capture the amplitude of the unique mode of the distribution. With $2k=2\cdot3=6$ moments, the reconstructed density correctly reproduce the right tail's shape but not the mode's amplitude. Only with over 20 moments the methodology is able to model the spike in the mode.

Table 7 – Reconstruction Analysis for the Index Linked Certificate

Original Cumulative Distribution Function and CDF Upper and Lower Bounds	*Original probability density function and 2k-moments reconstructions*
	

Degree of $P_k(x)$	Roots of $P_k(x)$	Lower bound $L_{k,i}(F)$	CDF $F(x)$	Upper bound $U_{k,i}(F)$
$k = 12$	$x_{12,1} = 7.771378093$	7.85215670e-09	7.325974324e-05	0.000366312
	$x_{12,2} = 18.39084429$	0.000366320	0.003885495	0.010731857
	$x_{12,3} = 30.47357083$	0.010731864	0.031274510	0.057394119
	$x_{12,4} = 45.18139610$	0.057394127	0.104401318	0.152774493
	$x_{12,5} = 61.04096629$	0.152774501	0.208148171	0.245887008
	$x_{12,6} = 82.26552865$	0.245887015	0.263709189	0.275108760
	$x_{12,7} = 111.4433923$	0.275108767	0.431189544	0.662323626
	$x_{12,8} = 117.9708981$	0.662323634	0.845583607	0.884228419
	$x_{12,9} = 160.0692863$	0.884228427	0.901194216	0.932902293
	$x_{12,10} = 166.4361793$	0.932902301	0.980268877	0.999616890
	$x_{12,11} = 184.2432208$	0.999616898	0.999817388	0.999931040
	$x_{12,12} = 198.2358246$	0.999931048	0.999967968	0.999999992

The probability density of the product is characterized by a complex morphology, with 3 distinct modes of changing amplitude. 6 moments are not enough to distinguish correctly the multimodality, that appears clearly only with 12 moments; with this level of detail, the amplitude of the 3 modes is not modelled properly. Better results are obtained with over 20 moments, but the spike of the main mode (the central one) cannot be reproduced even with more than 40 moments.

4.2 The optimal amount of information retrievable from moments

The consequential analysis should try to establish a quantitative indicator to measure the "goodness" of reconstruction and the gain in accuracy when more moments become available. The intuition suggests that the marginal benefit of adding more moments for the reconstruction should be decreasing, having considered the numerical instability that often arises when higher order moments are calculated and used to build the Christoffel polynomials. In this perspective, a sort of "optimal" number of moments should be identified in accordance with our empirical findings that show how simple and symmetrical distributions need fewer moments to be adequately reconstructed.

Subsequently, the classic Kolmogorov-Smirnov test to evaluate the identity between distributions has been re-adapted in order to determine how much of the original distribution is possible to reconstruct with the availability of a given number of moments.

In formal terms, the test statistics of the Kolmogorv-Smirnov test[18], i.e.

$$KS(k) = \max \left| F(x) - \tilde{F}_k(x) \right| \tag{23}$$

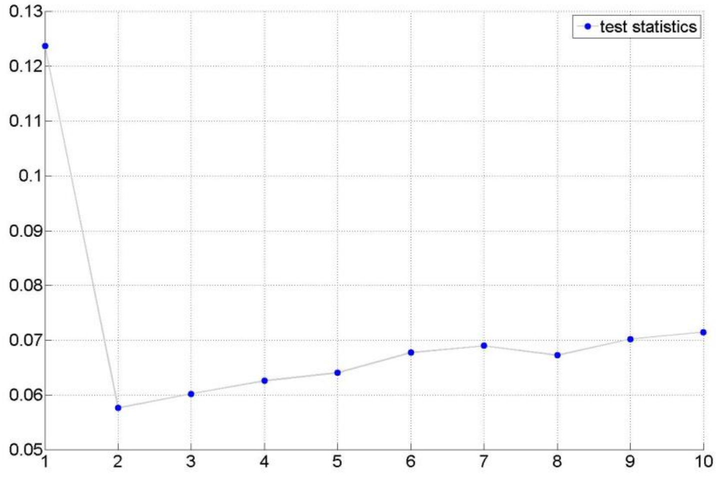
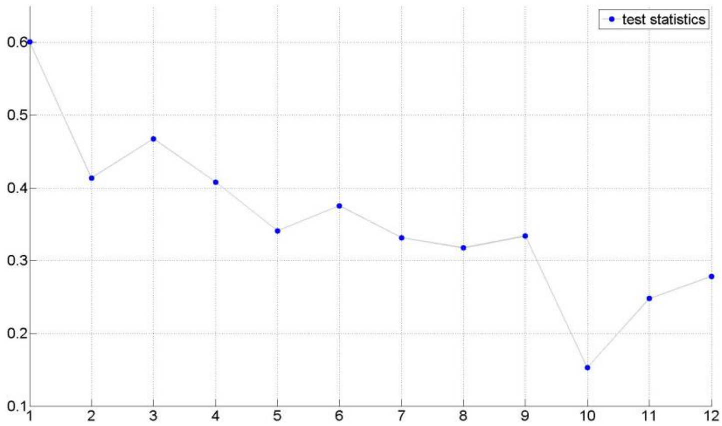
is calculated for each financial product, where $F(x)$ is the original cumulative probability distribution and $\tilde{F}_k(x)$ is the reconstruction that exploits exactly $2k$ moments. If the accuracy of the reconstruction improves when the number of moments increases, then the value of the statistics $KS(k)$ has to diminish steadily. Intuitively, in a plot of the statistics versus the parameter k, we should expect a decreasing function that converges by some power law to zero, since the Gavriliadis, Athanassoulis (2009) approximation (see equation 21) is asymptotically equivalent to the original cumulative probability distribution.

In reality, when k becomes big enough, numerical problems connected with the explicit calculation of the moments distribution emerge, and the $KS(k)$ statistics diverges rapidly to infinity. So, it can be stated that there exists a sort of "empirical optimum" number of moments that allows to obtain the best possible reconstruction with the described methodology.

The results obtained (see Table 7) shows unequivocally that simpler shapes are better approximated by this method and it's possible to achieve satisfactory estimates with very few moments; complex morphologies need a significant amount of moments and tends to produce less accurate reconstructions for the probability densities.

Table 8 – Results of the Modified Kolmogorov-Smirnov test for the 4 different financial products

[18] Abramowitz, M. and Stegun, I. A. 1957, Handbook of Mathematical Functions.

Low risk floater coupon bond *The chart clearly shows that the best reconstruction is obtained with only 4 moments.* *Adding information from the higher order moments paradoxically worsens the accuracy of the reconstruction, since we are adding unnecessary oscillations to the probability densities due to the polynomials behaviour; these oscillations fail to capture the unimodality of the density. Moreover, the benefit connected with a better modelling of the left tail are not sufficient to compensate the loss in accuracy.*	*Value of the Kolmogorov-Smirnov statistics* *(Original vs 2k-moments reconstructions)* 
Defaultable fixed coupon bond *From the analysis of the chart it can be inferred that adding information from moments improves the accuracy of the reconstruction up to 20 moments, when numerical instability sets in and the quality of the reconstruction begins to worsen.* *The level of accuracy reached with 20 moments is less than that obtained with 4 moments in the case of the low risk floater bond. This is related obviously to the more complex pattern of the probability density characterized by strong bimodality.*	*Value of the Kolmogorov-Smirnov statistics* *(Original vs 2k-moments reconstructions)* 

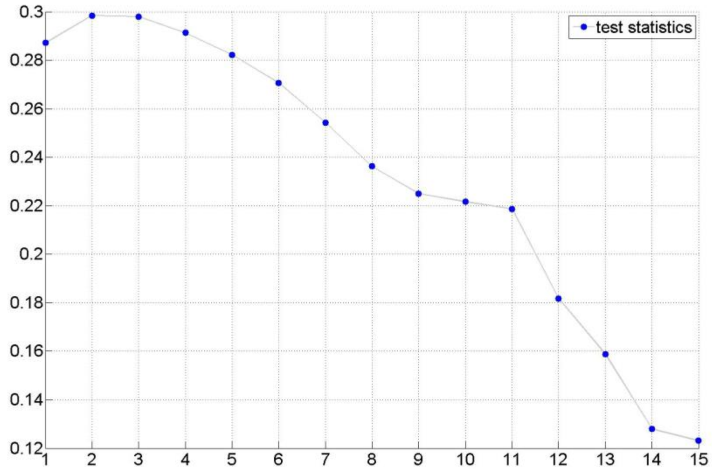
Variable Protection Portfolio Insurance

In this case, the long right tail that characterize the density is better captured with a fairly high number of moments. The results show that up to 28 moments the accuracy of the reconstruction continues to improve steadily.

The analysis is then stopped at 30 moments when it can be appreciated a flattening of the function, that is signalling a decreasing marginal benefit.

Value of the Kolmogorov-Smirnov statistics

(Original vs 2k-moments reconstructions)



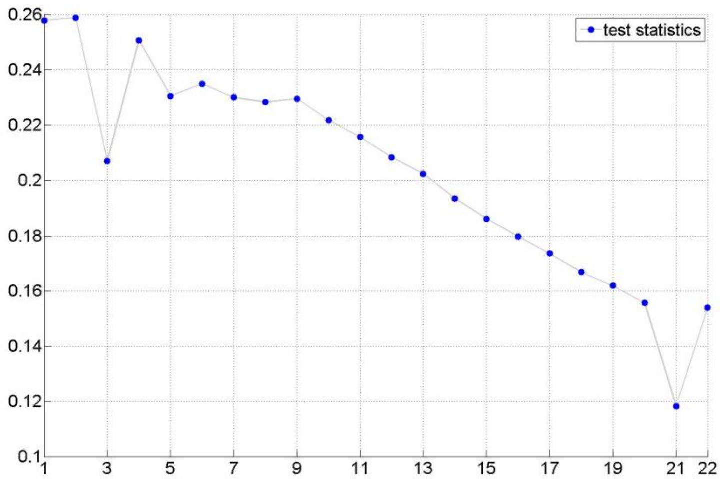
Index Linked Certificate

The difficulty of the methodology to correctly reproduce this multimodal complex morphology is evident in the significant number of moments needed to obtain a satisfactory reconstruction of the density and in the regular, linear decrease of the KS statistics observed up to 40 moments.

For higher order approximations, numerical instability sets in and so the analysis is stopped. The lowest level of error that we are forced to accept with 42 moments is more than twice of what we obtain in the case of the low risk floater bond.

Value of the Kolmogorov-Smirnov statistics

(Original vs 2k-moments reconstructions)



From test performed it has clearly emerged the correlation between a low quality of reconstruction of the original distribution and the significant divergence between the indicators related to the first few moments of the distribution. In other terms, a strong divergence in absolute terms is always signal of a poor significance of the first moments and of the presence of a complex distributional pattern. Vice versa, if the first moments are sufficient, there's a good chance of observing regular distribution characterized by a low dispersion, for which even the information conveyed by measures based on the average as the expected IRR or the fair value can be considered adequate.

In three cases of four, the reconstruction analysis clearly showed the need to add further information to the price and volatility in order to support the decisions of the average investor. In this perspective, it is essential that this additional set of information is alike understandable and intuitive.

4.3 The probabilistic performance scenarios

The most natural solution from a statistical point of view to the problem of extending the information set to the investor would be to illustrate a certain number of moments of higher order, till to obtain a satisfactory description of the features of the original distribution.

However, it is reasonable to doubt that an investor who does not have a precise statistical knowledge would be able to understand similar technical indicators. In fact, the probability distribution of the financial product and all the indicators of descriptive statistics represent abstract objects for the average investor, that have no immediate connection with his financial culture and with the risk/return description of the product.

The probabilistic performance scenarios proposed in section 3.2 could be a reasonable compromise that exploit the information contained inside the probability distribution of the PRIIP, combining the synthesis of moments with an adequate level of financial intuition.

For the calculation of the probabilities reported in Table 9[19], α_1 and α_2 have been chosen equal to 2.5 and 97.5 respectively. The mean values are calculated by using the formulas reported in Table 2.

Table 9 – Probabilistic performance scenarios for 3 different financial products

Defaultable fixed coupon bond

SCENARIOS	PROBABILITY	MEAN VALUES
The performance is negative	22.77%	56.10
The performance is positive but lower than the risk-free asset	0.00%	-
The performance is positive and in line with the risk-free asset	29.02%	127.61
The performance is positive and higher than the risk-free asset	48.21%	128.74

Variable Protection Portfolio Insurance

SCENARIOS	PROBABILITY	MEAN VALUES
The performance is negative	36.85%	97.46
The performance is positive but lower than the risk-free asset	18.54%	101.04
The performance is positive and in line with the risk-free asset	39.91%	104.59
The performance is positive and higher than the risk-free asset	4.70%	168.61

[19] Scenarios for the low risk floater bond have not been calculated since the probability density function is too similar to the risk-free asset used in the methodology as a benchmark. Accordingly, the results would have been not significant.

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SCENARIOS	PROBABILITY	MEAN VALUES
The performance is negative	27.21%	50.99
The performance is positive but lower than the risk-free asset	0.1%	100.87
The performance is positive and in line with the risk-free asset	60.94%	113.76
The performance is positive and higher than the risk-free asset	11.75%	163.72

Please note that in terms of risky components (table 2) the Defaultable fixed coupon bond has the greater value but in terms of probability scenarios (table 9) the higher risk (i.e. lower value in the downside case of negative performance) is carried by the Index linked certificate.

5. Discussion

Currently, no-arbitrage probability scenarios are widely used by the financial industry as risk management tools. Value-at-risk (VaR) is just an application to exploit part of the full information given by the market implied probability distribution[20].

Pitfalls with the use of no-arbitrage distributions (risk-neutral) for scenarios are a well-known issue in the literature. Already in the '90s Grundy (1991) was underlining that no-arbitrage probabilities were not "real-world" probabilities; by comparing two different estimates he tried to infer a measure of investors' risk aversion. Important progress has been made in the search of a better methodology in the last decade[21], but at the present time there's still no an established alternative.

Prior to the seminal work of Ross (2014), many efforts (Bliss and Panigirtzoglou 2004. Liu et al.2007, Humphreys and Noss 2012) were devoted to the task of extracting information about real-world probabilities in order to properly represents the expectations of risk-adverse investors. This stream of literature obviously intersected heavily with the ample research around the estimate of the risk premium, both from historical data (Siegel 1994, Jackwerth and Rubinstein 1996, Jackwerth 2000, Fama and French 2002, Dimson et al. 2003) or by calibrating equilibrium models of expected returns (Fama and French 2004, Campbell and Vuolteenhao 2004, Ang and Chen 2007, Hou et al. 2011, Cochrane 2011, Bollerslev and Todorov 2011).

In 2014 Ross introduced a new framework (the so called Recovery Theorem) to disentangle the information related to the risk premium from market prices under very specific technical assumptions on the stochastic

[20] Giving the expected loss corresponding to a percentile of the probability distribution of the value of the financial product at a future date, VaR requires always the estimation of the entire probability distribution of the value of a financial product at a given future date.

[21] For a balanced summary and a broader perspective on the subject of going beyond the no-arbitrage evaluation framework, see Brigo D. (2018), "Time to move on from risk-neutral valuation?", RISK. Link: https://www.risk.net/comment/5406771/time-to-move-on-from-risk-neutral-valuation.

process governing the underlying (univariate Markov process as an irreducible time-homogeneous finite-state Markov Chain and a measure change transition independent) that poses bounds to investors preferences[22]. This would imply a unique, non-subjective estimate of real-world probabilities that could be calculated without further hypotheses on risk premia.

The Ross result has given birth to two new brand streams of research: one devoted to the generalization of the theoretical result (Linetsky and Qin 2016, Qin and Linetsky 2017, Jensen et al. 2018, Walden 2019) and a second focused on the empirical implementation of the theorem (Audrino et al. 2014, Bakshi et al. 2015, Martin and Ross 2018). As of today, it is widely recognized that the Ross assumptions exclude a vast class of models from its range of applicability (Borovička et al. 2016).

For what regards the presentation of probabilistic results to the retail investor, early consumer tests[23] have shown the poor understandability of the entire probability distribution plotted in a chart as an histogram, even if the tests were partially flawed by design. Final results were suggesting the use of percentage instead of absolute numbers and of tables instead of histograms.

Recently, more encouraging results[24] are coming from the use of a table where probabilistic information is framed as "XX in 10 chance of doing worse" than the performance scenario. According to the most recent consumer tests, in terms of comparability "*the probabilistic approach version of the KID improved the identification of the products based on their specific features when comparing products of the same type or of different types. The probabilistic approach version of the KID helped participants identify the product with the most unpredictable returns when comparing a fund and an IBIP* [Insurance-Based Investment Products, Author's note] *and the product with the highest expected returns when comparing two IBIPs*". Moreover "*probabilistic information seemed to aid the participants' understanding of the probability of losses when comparing funds with IBIPs and structured products with IBIPs*". In general, "*the probabilistic approach (with or without additional information on past performance or an illustrative scenario) generally had a positive influence on understandability for funds and IBIPs*".

6. Conclusions

In the present work we offered a new way to represent probabilistic performance scenarios for PRIIPs, starting from the concept of implied probability of a financial product to develop an alternative representation of the information contained in the probability distributions w.r.t. the solutions currently in evaluation by the ESAs. The proposal has the aim to be at the same time effective in conveying the key statistical features of the PRIIP and easily understandable by the average investor.

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[22] In particular, Ross introduced the concept of *state prices* (market prices of contingent forward contracts). In other words, in order to apply the Recovery Theorem, one should have a complete knowledge of the prices of these securities prices at future dates conditional on being in any other state of world . If the preferences of the market were correctly aggregated - he assumed the existence of a representative agent -then they would only depend on the final state, the initial state and the knowledge of the path would not be necessary.

[23] See footnote 11.

[24] FISMA/2019/016/C , "Consumer testing services -Retail investors' preferred option regarding performance scenarios and past performance information within the Key Information Document under the PRIIPs framework", Executive Summary. Link: https://ec.europa.eu/info/sites/info/files/business_economy_euro/banking_and_finance/documents/200227-consumer-testing-services-summary_en.pdf.

This representation can overcome the weakness of the approach currently in the testing phase, before it is extended to all retail investment products. This is true especially for the PRIIPs that are characterized by a structured financial engineering and by a complex pattern of the implied probability distribution. In those cases, in fact, the PRIIP embeds several risks that could not be adequately perceived by the investor by the means of performance scenarios that are representative of a single sample of the probability distribution.

Statistical tests of reconstruction for the probability densities show that in many realistic cases a huge number of moments would be required to adequately capture the main features of the probability distribution of future values. Nevertheless, it is suggested that the description of quantiles connected with events of financial interests for the investor can suitably substitute the higher order moments without loss of relevant information.

References

Ang A. and J. S. Chen (2007), "CAPM Over the Long-Run: 1926-2001", Journal of Empirical Finance, vol. 14, Issue 1, 1-40.

Audrino F., Huitema R. and Ludwig M. (2014), "An empirical analysis of the Ross recovery theorem", Economics Working Paper Series no.1411, University of St. Gallen.

Bakshi G., Chabi-Yo F., Gao X. (2015). "A recovery that we can trust? Deducing and testing the restrictions of the recovery theorem", The Review of Financial Studies 31, 532-555.

Breeden, Douglas and Litzenberger, Robert H. (1978), "Price of State-contingent Claims Implicit in Option Prices", Journal of Business 51, pp 621-651.

Black, F. and Scholes, M. (1973), "The Pricing of Options and Corporate Liabilities", Journal of Political Economy 81 (3), 637-654.

Bliss and Panigirtzoglou (2004), "Option implied risk-aversion estimates", Journal of Finance, Vol. 59, Issue 1.

Bollerslev T., Todorov V., (2011). "Tails, fears, and risk premia", Journal of Finance 66, 2165-2211.

Borovicka J., Hansen L.P., Scheinkman J.A. (2016) "Misspecified Recovery", Journal of Finance, Vol. 71, Issue 6.

Campbell, J.Y. and Vuolteenaho T. (2004), "Bad Beta, Good Beta", American Economic Review, vol. 94, no. 5.

Cochrane, J. (2011), "Presidential Address: Discount Rates", Journal of Finance, Vol. 66 Issue 4.

Commission Delegated Regulation (EU), of 30 June 2016, supplementing the Regulation (EU) n. 1286/2014.

Commission Delegated Regulation (EU), 2017/653 of 8 March 2017, supplementing the Regulation (EU) n. 1286/2014.13.

Commission Delegated Regulation (EU), 2017/653 of 8 March 2017, supplementing the Regulation (EU) n. 1286/20.

Cornish, E. A. and Fisher, R. A. (1938). "Moments and Cumulants in the Specification of Distributions" (PDF). Revue de l'Institut International de Statistique / Review of the International Statistical Institute. 5 (4): 307–320.

Dimson E., Marsh P. and Staunton M. (2003), Global Evidence on the Equity Risk Premium, Journal of Applied Corporate Finance, Vol. 15 Issue 4.

Duffie D., and Pan J., “An Overview of Value at Risk”, Journal of Derivatives 4 (Spring), 7-49, 1997.

European Commission (2014), “Consumer testing study of the possible new format and content for retail disclosures of packaged retail and insurance-based investment products”, Final Report by London Economics and IPSOS, European Union, MARKT/2014/060/G for the implementation of the Framework Contract n° EAHC-2011-CP-01. Link: https://ec.europa.eu/info/sites/info/files/consumer-testing-study-2015_en.pdf.

ESMA (2020), “Annex to Letter ESA 2020 19-Draft Final Report following consultation on draft regulatory technical standards to amend the PRIIPs KID”. Link: https://www.esma.europa.eu/sites/default/files/library/jc_2020_66_priips_rts_draft_final_report.pdf.

Fama E. F. and K. R. French (2002), “The Equity Premium”, Journal of Finance, Vol. 57 Issue 2.

Fama E.F. and French K.R. (2003), “The Capital Asset Pricing Model: Theory and Evidence”, Journal of Economic Perspectives, Vol. 18, Number 3.

FISMA/2019/016/C, “Consumer testing services -Retail investors’ preferred option regarding performance scenarios and past performance information within the Key Information Document under the PRIIPs framework”. Link: https://ec.europa.eu/info/sites/info/files/business_economy_euro/banking_and_finance/documents/200227-consumer-testing-services-summary_en.pdf.

Gavriliadis P. N., Athanassoulis G. A. (2009), “Moment information for probability distributions, without solving the moment problem II: Main-mass, tails and shape approximation”, Journal of Computational and Applied Mathematics, pp. 7-15.

Grundy, B. D. (1991), Option prices and the underlying asset’s return distribution, Journal of Finance.

Hou K., Karolyi G. A. and Kho B. (2011), “What Factors Drive Global Stock Returns?”, Review of Financial Studies, vol. 24, issue 8.

Humphreys V. and J. Noss (2012), “Estimating probability distributions of future asset prices: empirical transformations from option-implied risk-neutral to real-world density functions”, Bank of England, Working Paper.

Jackwerth J. (2000), “Recovering risk aversion from option prices and realized returns, Review of Financial Studies” 13, 433-451.

Jackwerth J. C., Rubinstein, M. (1996), “Recovering probability distributions from option prices”, Journal of Finance 51, 1611-1631.

Jensen C. S., Lando D., Pedersen L.H (2019), “Generalized Recovery”, Journal of Financial Economics, 133 (1), 154-174.

Linetsky V. and Qin L., (2016), “Positive eigenfunctions of markovian pricing operators: Hansen-Scheinkman factorization, Ross recovery, and long-term pricing”, Operations Research 64, 99-117.

Liu, X., M. Shackleton, S. Taylor and X. Xu (2007), Closed-form transformations from risk-neutral to real-world distributions, Journal of Banking and Finance.

Martin I. and Ross S. (2018), “Notes on the yield curve”, Journal of Financial Economics 134 689–702.

Minenna M. et al. (2009) “A quantitative risk-based approach to the transparency on non-equity investment products”, Quaderno di Finanza n.63, Consob.

Minenna M. et al., (2011) “A Quantitative Framework to Assess the Risk-Reward Profile of Non-Equity Products”, Risk Books, pp. 26-30.

Qin L. and Linetsky V. (2017), “Long-term risk: a martingale approach”, Econometrica 85, 299-312.

Ross, S. (2014), “The Recovery Theorem”, The Journal of Finance 70, Issue 2.

Siegel, J. (1994), “Stocks for the Long Run”, Irwin Professional Publishing.

Shohat J. A., Tamarkin J. D. (1943), “The Problems of Moments”, American Mathematical Society.

Szego, G. (1959), “Orthogonal Polynomials”, vol.23, American Mathematical Society.

Totik, V. (2000), “Asymptotics for Christoffel functions for general measures on the real line”, Journal d’Analisè Mathèmatique, n.81.

Walden J. (2017), “Recovery with unbounded diffusion processes”, Review of Finance 21, 1403-1444.

Annex

Technical details about the four theoretical financial products presented in the paper

<table>
<tr><td>Low risk floater coupon bond</td><td>Models: Interest Rate: 2 Factor Hull-White

Calibration Date: November 3, 2010.

Pay-Off: Slight modification of a basic Cash Account that includes fees.

No. Of Simulations: 50.000</td></tr>
<tr><td>Defaultable fixed coupon bond</td><td>Models: Interest Rate: 2 Factor Hull-White

Credit Risk: estimated from a Credit Default Swap quoted at 332.5 bps

Recovery Rate given Default: 40%

Calibration Date: November 3, 2010.

Pay-Off: Fixed Rate (5.42%) paid twice a year.

No. Of Simulations: 50.000</td></tr>
<tr><td>Variable Protection Portfolio Insurance</td><td>Models: Interest Rate: 2 Factor Hull-White. Risky Portfolio: Geometric Brownian Motion with 30% volatility.

Calibration Date: November 3, 2010.

Pay-Off: Dynamic rebalancing between a zero coupon and a risky share of an overall portfolio with initial invested value of 93. The weights depends from the size of a “cushion”, i.e. the difference between the current value of the investment and the zero-coupon that is able to achieve 100 at maturity.

The cushion is multiplied against a leverage coefficient and then the calculated amount is invested in the risky asset. The remaining capital is invested in the zero coupon. The leverage coefficient is a stochastic variable that is path-dependent whose size is determined by looking at the cushion size.

<table><tr><td>Cushion size</td><td>Leverage coefficient</td></tr><tr><td>Between 0 and 5</td><td>1</td></tr><tr><td>Between 5 and 20</td><td>2</td></tr><tr><td>Over 20</td><td>3</td></tr></table>
As an example: at the start of the simulation the Net Asset Value (NAV) is 98 due to 2 of one-off fees. The zero coupon then worth 88 and the cushion has the value of 10. According to the above table, the leverage coefficient is 2.

The weights of the two assets classes are hence: 20 (2*10) for the risky asset while the remaining capital (i.e. 98-20=78) is invested in the zero-coupon.</td></tr>
</table>

	By looking at the random daily variations of the risky assets, the weights are dynamically adjusted. Fees: Initial 7 (one-off), recurring 1 (paid yearly). No. Of Simulations: 50.000.
Index Linked Certificate	Models: Interest Rate: 2 Factor Hull-White. Risky Portfolio: Geometric Brownian Motion with 30% volatility. Calibration Date: November 3, 2010. Pay-Off: Variable coupon, paid twice yearly. The value of the periodic coupon is calculated by considering a 5% fixed rate minus the Euribor rate 6 months plus a spread of 30 bps. The underlying is a risky asset with a constant volatility of 30%. There are two relevant barriers: low_barrier = 0.50 * S0 high_barrier = 2 * S0 At settlement dates, the risky asset is evaluated against the barriers: if the lower one is trespassed, the index certificate is early terminated and only a fraction of the invested capital is paid back (less than half). Formally: FinalValue = FinalValue + CN*SPaths(end)/S0 where: CN is the invested capital, Spaths(end) is the last path's value of the simulated risky asset and S0 is the initial value of the risky asset. If the former event does not happen during the life of the product, at maturity the entire invested capital is reimbursed, i.e. CN=100. At maturity, the investor can get the chance of an extra-coupon: the risky asset is evaluated against the high barrier: if it is trespassed, a coupon equal to half of the invested capital is paid. Fees: Initial 8 (one-off). No. Of Simulations: 50.000.